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A Study of Monotone Variational Inequalities and their Application to Network Equilibrium Problems

by
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To my father

my mother

my wife

my son

Summary

In this thesis, we consider the finite monotone variational inequality problem VIP in a systematic way. VIP consists in finding a solution $v^* \in \Omega \subset \Re^m$ which satisfies $s(v^*)^T(v - v^*) \geq 0$, for all $v \in \Omega$. The current state of research into theory and algorithms in this field is reviewed. The VIP plays a significant role in many applications such as in economics and transportation in addition to several academic fields. An equivalent optimization problem of VIP is formulated. Based on it, a general descent framework for the problem is established with a proof of its convergence. Several realizations of the framework are also discussed to indicate its relationship to some existing algorithms. An inexact descent algorithm is also proposed. A similar analysis is applied to a transformed monotone variational inequality problem VIP_h where the cost function is only monotone. The developed framework then is applied to the network equilibrium problem where the cost structure is asymmetric. Some specific algorithms are presented with respect to the network structure. In what follows, we establish a model for transit equilibrium assignment. This model is then formulated as VIP_h and is solved with both a linearized Jacobi method and a projection method. Finally, numerical results are reported. The conclusions are given with possible research opportunities and general comments.

In short, the contributions of the thesis are:

1. A literature review of the VIP theory and algorithms is given;
2. A general descent framework for the VIP equivalent optimization problem is established with a detailed analysis;
3. An inexact method is developed for the VIP;
4. A global convergence proof is given for the linearized Jacobi method for VIP_h;
5. A similar analysis is applied to the VIP_h;
6. An efficient algorithm is developed for a special quadratic program;

7. A model for the transit equilibrium assignment is established and formulated as VIP_h and is solved with the linearized Jacobi method.

Key words: Variational inequality problem. Optimization. Network equilibrium. Traffic equilibrium. Transit equilibrium. Mathematical programming. Linearized Jacobi method. Projection method.

Résumé

Dans cette thèse, nous étudions le problème d'inégalité variationnelle (PIV), qui consiste à trouver une solution $v^* \in \Omega \subset R^m$ au système d'inégalités

$$s(v^*)^T(v - v^*) \geq 0, \quad \forall v \in \Omega.$$

Nous résumons d'abord l'état de la recherche théorique et algorithmique dans ce domaine. Nous rappelons également que le PIV joue un rôle important dans de nombreux champs d'études, notamment en économie et dans le domaine des transports.

Nous considérons ensuite le problème d'optimisation suivant:

$$\min_v f(v) = -s(v)^T(H(v) - v) - \frac{1}{\alpha}\phi(H(v), v)$$

$$\text{où. } H(v) = \arg\min_u s(v)^T(u - v) + \tfrac{1}{\alpha}\phi(u, v)$$

$$\text{s.à. } u \in \Omega$$

et $\phi(u, v) : \Omega \times \Omega \subset R^m \to R^1$ possède les propriétés suivantes:

- ϕ est continûment différentiable sur $\Omega \times \Omega$;
- ϕ est non-négative sur $\Omega \times \Omega$;
- ϕ est strictement convexe sur Ω;
- $\phi(u, v) = 0 \Leftrightarrow u = v$;
- α est un paramètre positif.

Notons que si $\alpha = \infty$, alors $H(v)$ peut posséder un ensemble comme image.

Nous démontrons que ce problème est équivalent au PIV dans le sens que $f(v) \geq 0$, $\forall v \in \Omega$ et $f(v^*) = 0$ si et seulement si v^* est une solution du PIV. Si v n'est pas une solution du PIV, alors, sous certaines conditions, il existe $u \in H(v)$ tel que $d = u - v$ est une direction de descente pour f en v. Dans certains cas, que nous traitons en détails, un point stationnaire v^* de f est associé à solution

du PIV. De plus, nous étudions plusieurs formes possibles pour la fonction $\phi(u,v)$ et nous donnons une méthode permettant de la construire. Pour résoudre le problème d'optimisation, nous introduisons ensuite une méthode générale de descente pour laquelle nous fournissons une preuve de convergence globale. Nous démontrons également que le PIV peut être reformulé à la fois comme un problème de "max-min" et comme un problème de programmation biniveau. Nous discutons ensuite des propriétés des multiples variantes de la méthode générale de descente: la méthode de Newton symétrique, la méthode Quasi-Newton, la méthode de Jacobi linéarisée, la méthode de projection, la variante de Migdala de la méthode de Frank et Wolfe (1990), la méthode de Dussault, Ferland et Lemaire (1986) et la méthode de Fukushima (1989).

Par la suite, nous étudions une variante de la méthode appelée méthode inexacte. Dans cette variante, le sous-problème est résolu à ϵ près par la méthode de Frank et Wolfe. Nous décrivons des conditions permettant d'identifier une direction de descente et nous établissons la convergence globale de la méthode. Une approche de décomposition peut être utilisée pour résoudre le sous-problème.

Dans cette thèse, nous considérons également le problème d'inégalité variationnelle transformé (PIV_h) qui consiste à trouver une solution $h^* \in \Omega_h \subset R^m$ au système d'inégalités

$$S(h^*)^T(h - h^*) \geq 0 \qquad \forall h \in \Omega_h$$

où $S(h) = \delta^T s(v), v = \delta h$ et δ est une matrice de transformation linéaire $m \times n$. Une difficulté particulière du PIV_h est que l'application $\nabla S(h) = \delta^T \nabla s(v)\delta$ peut être positive semi-définie. Dans ce cas, la théorie classique ne garantir par la convergence globale des méthodes de descente. En particulier, pour le problème d'équilibre de réseau considéré dans l'espace des flots de chemins, $\delta^T\delta$ est une matrice singulière. Nous généralisons la théorie développée pour le PIV et nous démontrons que même dans ce cas, la méthode que nous proposons est toujours une méthode de descente et qu'elle est globalement convergente. Nous discutons des liens entre les points stationnaires et les solutions du PIV_h et nous établissons les conditions permettant d'identifier une direction de descente. Par la suite, nous nous tournons vers une variante spéciale de la méthode, soit la méthode de Jacobi linéarisée et nous donnons une preuve de convergence globale, qui constitue un

des apports originaux de cette thèse. Nous insistons également sur les avantages de la méthode de Jacobi linéarisée par rapport à la méthode de projection.

En se basant sur la théorie développée, nous appliquons notre algorithme de descente au problème d'équilibre de réseau, qui consiste à trouver $h^* \in \Omega_h$, un flot d'équilibre dans l'espace des chemins, tel que

$$S(h^*)^T(h - h^*) \geq 0, \quad \forall h \in \Omega_h,$$

où

h est un flot réalisable;
$S(h^*)$ est le vecteur de coût correspondant au flot de chemin h^*;
Ω_h est la région réalisable définie de la façon suivante:

$$\sum_{k \in K_i} h_k = g_i, \quad \forall\ i \in I,$$

$$h_k \geq 0, \quad \forall\ k \in K_i, \quad i \in I,$$

I est l'ensemble des paires origine-destination;
K_i est l'ensemble des chemins associé à i;
h_k est le flot du k^e chemin;
g_i est la demande associée à la paire origine-destination i.

L'application de notre algorithme pour résoudre ce problème semble particulièrement prometteuse puisque le sous-problème est traité par une méthode de recherche uni-dimensionnelle que nous avons développé.

Une application importante de cette étude est le problème d'affectation d'équilibre dans un réseau de transport en commun ($PAET$), dont nous considérons la formulation et dont nous étudions les algorithmes de résolution déjà proposés. L'objectif du $PAET$ est de modéliser le flot des passagers sur un réseau de transport en commun, flot qui dépend d'une matrice de demande origine-destination. Le réseau de transport en commun qui permet de modéliser le problème est constitué du réseau des routes et des lignes de véhicules qui l'utilise. Notre approche de résolution de ce problème est basée sur le concept d'hyperchemin dans la formulation du $PAET$ et nous l'appliquons dans le cadre d'un réseau général représentant le réseau physique de transport en commun. Dans ce réseau, il y a quatre types d'arcs: les arcs de marche, les

arcs d'attente, les arcs à bord des véhicules, et les arcs de correspondance. Ce réseau général, où on associe à chaque arc une fréquence et un coût de transport, est une généralisation du réseau physique traditionnel, tel que défini par Spiess et Florian (1989). Notre formulation du coût d'attente est également différente de celles des modèles existants. Dans notre cas, le coût d'attente est la somme d'une fonction qui dépend des fréquences et d'une autre fonction qui dépend des flots d'attente et des flots de vehicules. La structure des fonctions de coût pour les quatre types d'arcs est étudiée. Les comportements des passagers sont formulés à l'aide du modèle du PIV_h. Des spécialisations de la méthode de Jacobi linéarisée et de la méthode de projection sont développées pour résoudre la $PAET$ et nous donnons également des preuves de convergence globale. Finalement des implantations de ces algorithmes sont réalisées. Il découle des tests effectués à partir de ces implantations que la méthode de Jacobi linéarisée fournit d'excellents résultats et converge après un petit nombre d'itérations, ce qui confirme notre étude théorique. Le nouveau modèle se comporte très bien, en particulier dans le cas d'un réseau de transport en commun congestionné.

En conclusion, nous pouvons résumer les contributions majeures de cette thèse:

1. Nous faisons une revue de la littérature sur le PIV, aussi bien au plan théorique qu'algorithmique;
2. Nous donnons une analyse détaillée de la méthode générale de descente pour résoudre un problème d'optimisation équivalent au PIV;
3. Nous développons une méthode inexacte pour résoudre le PIV;
4. Nous donnons une preuve de convergence globale pour la méthode de Jacobi linéarisée appliquée au PIV_h;
5. Nous développons une analyse similaire pour le PIV_h;
6. Nous développons un algorithme efficace dans le cas d'un problème particulier, celui d'un programme quadratique;

7. Nous développons un modèle pour le $PAET$ et il est formulé comme un PIV_h et résolu par une méthode de Jacobi linéarisée et une méthode de projection.

D'autres avenues de recherche sont également étudiées.

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List of Abbreviations

1. LJBOM - Linearized Jacobi based optimization method
2. LJM - Linearized Jacobi method
3. PBOM - Projection based optimization method
4. PM - Projection method
5. $Proj_{\Omega,B}(u)$ - a projection of a point u with respect to the region Ω and B-norm
6. $TEAP$ - the transit equilibrium assignment problem
7. u.s.c. - upper semicontinuous
8. VIP - the variational inequality problem
9. VIP_h - the transformed variational inequality problem
10. $VIP(\Omega, s)$ - the variational inequality problem defined in the region Ω with a function s
11. $VIP_h(\Omega_h, S)$ - the transformed variational inequality problem defined in the region Ω_h with a function S

Chapter 1

Introduction

1.1 The scope of the thesis

In this thesis, we look at and study the finite-dimensional monotone variational inequality problem VIP in terms of the theory and algorithms. Some other subjects with close relationship with VIP are omitted such as the sensitivity analysis (Harker and Pang, 1990), the optimization with VIP as constraints (Friesz et al., 1990), the dynamic traffic assignment (Drissi-Kaitouni, 1990) and the infinite-dimension variational inequality problem. Our emphasis is on the reformulations of VIP, the algorithmic developments and their applications in transportation such as the network equilibrium and transit equilibrium assignment, though the concept is very general and can be extended to many other areas.

1.2 The historical review

This historical review has been largely influenced by the survey paper of Harker and Pang (1990). There are several important periods in the development of the VIP. Firstly, Lemke and Howson (1964) and Herbert Scarf (1967) initiated the great interest in the economic and game theoretic equilibria in the academic and professional communities. The initial methods for computing equilibria were all based on the constructive proof by Lemke and Howson of the existence of an equilibrium point for a bimatrix game. These methods have become

known as fixed-point methods. Many well-known pioneers in this field were: Scarf (1967) who developed the first algorithm to approximate a fixed point of a continuous mapping based on concepts of primitive sets, Hansen (1968) who introduced simplexes, Eaves (1976, 1978) who introduced piecewise-linear maps into the computational fixed-points. These methods have been used in many applications such as taxation and unemployment (Manne, 1985, McKenzie, 1976, Shoven, 1977). However, the fixed point methods have experienced difficulty in solving medium to large-scale equilibrium models, as evidenced by the the PIES (Project Independence Evaluation System) energy model (Ahn, 1979) which was developed at the U.S. Department of Energy in the late 1970's.

Secondly, there have been growth in the use of the equilibrium models in the area such as transportation and strategic planning, which required to develop efficient methods for computational analysis. The first contribution to this subject was due to Beckmann et al. (1956). It is interesting to note that his work was reconsidered and extended after many years by the researchers in the transportation field. Computing the traffic equilibrium is very important to the study of transportation planning such as transportation alternative evaluations, which motivates many research (see Dafermos, 1971, 1972, LeBlanc, 1975, Florian, 1977, Florian and Nguyen, 1978). The most popular method for solving the traffic equilibrium problem (network equilibrium problem) is based on the Frank-Wolfe method by LeBlanc et al. (1975). Garcia and Zangwill (1981) developed a nonlinear optimization (symmetric equilibrium programming or symmetric optimization) for solving the equilibrium models. However, this kind of approaches imposes very restrictive assumptions on the model, which may not be true in real applications. Therefore it does not provide a satisfactory alternative to fixed point methods.

Thirdly, the VIP has been proven to be equivalent to the network equilibrium problem (Smith, 1979, and Dafermos, 1980). The solution procedures for the VIP have developed very rapidly and are considered very promising approaches as compared with the fixed point methods and the symmetric optimization methods. There are three

major events in the development of algorithms to be noted. The first one is the PIES project which provided an important contribution to the problem. The second event is due to the results of M.J. Smith (1979), S.Dafermos (1980) and Fang (1982 [71] [72]) who established the relationship between the traffic assignment problem and a variational inequality problem. Inspired by their work, many models have appeared in practice; see Friesz, Tobin, Smith and Harker (1984) for the prediction of interregional commodity flows, Gabay and Moulin (1980) and Harker (1984 [109]) for the solution of Nash equilibria and Harker (1982) for the estimation for freight transportation cost. The last event is due to Lars Mathiesen who solved the Walrasian or general equilibrium by nonlinear complementarity problem with encouraging results. However, his method is not proven convergent. Furthermore, an interesting comparison between the fixed point methods and the nonlinear complementarity method was carried out by Preckel (1985), which indicated the weakness of the former method in practice. Cohen (1988) proposed an extension of the linear approximation algorithm by Pang and Chan (1982) using the auxiliary problem principle. For a detailed literature review, see Harker and Pang (1990) who give a comprehensive, state-of-the-art review of the finite-dimensional variational inequality and the nonlinear complementarity problems.

Recently, much interest has been dedicated toward the reformulation of the VIP as an optimization problem where the Jacobian matrix of the cost function is not restricted to be symmetric, thus yielding an approach which is more general than the traditional optimization approaches. This kind of VIP is called the asymmetric variational inequality problem. One motivation of having such formulations is that powerful optimization tools can be utilized for solving them. See Hearn (1982), Smith (1983), Hearn, Lawphongpanich and Nguyen (1984). Nguyen and Dupuis (1984) proposed a cutting plane approach for an equivalent but nondifferentiable formulation. Marcotte (1985) proposed a descent algorithm for an equivalent nonconvex and nondifferentiable optimization problem for monotone function, while Dussault and Marcotte (1985) solved an equivalent program. A significant contribution to the reformulation as an equivalent optimization problem is that of Fukushima (1989) who developed a continuously

differentiable and equivalent optimization model based on a projection operator. This method is called projection based optimization method (PBOM).

Pang (1990) used a theory of a B-differentiable system of nonlinear equations to develop an equivalent optimization formulation which is B-differentiable and solves it using Newton's method.

An important observation is that there are few studies in which the relationships among these solution methods for the VIP are developed, evaluated and generalized. This is precisely our aim of conducting the research, where we develop a descent framework for a general formulation of the asymmetric variational inequality, which includes many existing algorithms such as projection methods, linearized Jacobi methods and Fukushima's method (1990).

1.3 Contributions of the thesis

The contributions of the thesis are as follows:

1. A literature review of theory on the VIP with emphasis on the asymmetrical reformulations and their solution methods;
2. A general optimization formulation of the VIP is established within a general descent framework; the proof of global convergence of the framework is given. The descent framework is so general that it, in fact, includes many existing algorithms.
3. An inexact method is developed for the VIP.
4. A global convergence proof is given to the linearized Jacobi method for VIP_h for which only local convergence proof is available.
5. The similar analysis is applied to a transformed VIP_h equivalent optimization problem where the cost function is only monotone for both the general descent framework and the linearized Jacobi method.
6. An efficient algorithm is developed for a special quadratic program.

7. A model for the transit equilibrium assignment is established, formulated as VIP_h and solved with a linearized Jacobi method and a projection method. The computational numerical results are also reported.

1.4 Chapter review

In the following, we give a brief review of each Chapter.

In Chapter Two, some basic notation and definitions are given. The VIP is stated with its transformed variational inequality problem. The preliminary mathematical concepts related to our presentation are also reviewed.

In Chapter Three, we present a literature review on the VIP in terms of its theoretical analysis (existence and uniqueness), various reformulations of the VIP and the solution approaches. However, the emphasis is on reformulations of the VIP and on solution approaches. A discussion is given of solution methods in quadratic programming.

In Chapter Four, we focus on the theoretical development of the VIP. One motivation is to establish a more general optimization formulation which covers many of the optimization formulations of VIP. Based on the formulation, a general descent framework is derived to provide a solution procedure for the VIP. A proof of the global convergence of the framework is given. Then various realizations of the framework are presented which indicate its relationship to other existing algorithms for both the VIP and optimization problems. Based on the framework, an inexact descent algorithm is proposed with a proof of its convergence under certain conditions.

In Chapter Five, we work on the theoretical development for VIP_h in parallel. First, we present some basic results special to the VIP_h. Then a general optimization formulation for VIP_h is given. A corresponding general descent framework is derived with a proof of its convergence. The linearized Jacobi method, a special case of the framework is stated with a global convergence proof. This method is then

compared with a variable projection method.

In Chapter Six, we apply the results in Chapter Five to the network equilibrium problem and develop a special algorithm for the problem. Finally, we give a special implementation of an algorithm for the strictly convex quadratic program.

In Chapter Seven, we mainly consider the transit equilibrium assignment problem ($TEAP$). A brief literature review of the transit assignment problem is presented. An important contribution of the model that we have is that it accounts for congested transit travel behaviour, which is not available currently. This problem is shown to be the VIP_A and is solved by the linearized Jacobi method and by the projection method. Promising numerical results are also reported.

In Chapter Eight, we provide a summary of our research and experiences. We mention some of the problems we have encountered, some of opportunities for further research and extensions of the present work.

Chapter 2

Problem Statement and Preliminaries

In this chapter, we first give the definition of the variational inequality problem in both standard and transformed formulations. We then give some mathematical preliminaries necessary for the presentation.

2.1 Variational inequality problem

The variational inequality problem VIP is defined as follows:
find $v^* \in \Omega$ such that

$$(VIP) \qquad s(v^*)^T(v - v^*) \geq 0, \quad \text{for all} \quad v \in \Omega \quad (2.1)$$

or alternatively let

$$v = \delta h, \qquad S(h) = \delta^T s(v) \qquad (2.2)$$

find $h^* \in \Omega_h$ such that

$$(VIP_h) \qquad s(\delta h^*)^T \delta(h - h^*) \geq 0, \quad \text{for all} \quad h \in \Omega_h (2.3)$$

where
$\Omega \subset R^m$ and $\Omega_h \subset R^n$ are nonempty sets;
$v \in R^m$ and $h \in R^n$ are both vectors ;
$s(v) : \Omega \subset R^m \to R^m$ is a mapping;

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$S(h) : \Omega_h \subset R^n \to R^n$ is a mapping;
δ is an $m \times n$ linear transformation matrix;
T denotes transposition.

We assume that $\Omega_h = \{h | \delta h = v\}$ is nonempty for all $v \in \Omega$. We call VIP the standard variational inequality problem and VIP_h the transformed variational inequality problem. In the following chapters, occasionally we use $VIP(\Omega, s)$ to define VIP on Ω with function $s(v)$ and $VIP_h(\Omega_h, S)$ to define VIP_h on Ω_h with function $S(h)$.

All the results stated below are applicable for VIP. We discuss VIP_h in Chapter 5. One typically assumes that the set Ω is closed and convex; in fact, Ω is often polyhedral in applications.

In general, if v^* is a solution of the VIP, then the angle between the vector $s(v^*)$ and all the feasible vectors emanating from v^* must be acute. Formally, $v^* \in \Omega$ is a solution to $VIP(\Omega, s)$ if and only if $s(v^*)$ is inward normal to Ω at v^*; i.e., as observed in Smith (1979), $-s(v^*)$ belongs to the normal cone $N_\Omega(v^*)$ where:

$$N_\Omega(v^*) = \begin{cases} \{z : (v - v^*)^T z \leq 0, \quad \forall \ v \in \Omega\} & \text{if } v^* \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

In the following, we provide two examples of the VIP. In the first example, we consider the VIP where $s(v)$ is an affine mapping, i.e., $s(v) = q + Mv$ and $\Omega = \{v : Av = b\}$. Then according to Rockafellar (1980), the VIP can be restated as a system of linear equations as follows: find (v, β) such that

$$Mv + q + A\beta = 0 \quad \text{and} \quad Av = b$$

are satisfied. If M is symmetric, then the system of linear equations is the Kuhn-Tucker condition for the minimization of a quadratic program.

The second example is that of differentiable optimization (Rockafellar, 1968, Mancino and Stampacchia, 1972). It is well known that the $VIP(\Omega, s)$ is a restatement of the necessary condition for v^* to be a local optimization of the optimization problem $\min_{v \in \Omega} f(v)$ if $s(v)$ is

the gradient of the real-valued differentiable function $f : R^m \to R$ and if Ω is convex.

In other words, v^* solves the problem $VIP(\Omega, s)$ if and only if v^* is a stationary point of $f(v)$. This means that any local optimal solution is a solution of the VIP which is a global optimal solution of the optimization problem, which is very important to observe.

The distinct relationship between the VIP and these optimization problems may not hold if s is not a gradient mapping, according to the symmetry principle (Ortega and Rheinboldt, 1970, Theorem 4.1.6). However there are certainly some other relationships between the VIP and optimization problems which are called asymmetric optimization problem. This will be discussed later as our main subject.

2.2 Preliminaries

Definition 2.2.1 *Let Ω be a nonempty, closed and convex subet of R^m and let B be any $m \times m$ symmetric positive definite matrix. Then the projection under the B-norm of a point $u \in R^m$ onto the set Ω, denoted as $Proj_{B,\Omega}(u)$, is defined as the solution (which must exist and be unique) to the following mathematical program:*

$$\min_{v \in \Omega} \|u - v\|_B^2 \tag{2.4}$$

where $\|x\|_B = (x^T B x)^{\frac{1}{2}}$ denotes the B-norm of a vector $x \in R^m$.

Moreover, for each $v \in R^m$, $Proj_{\Omega,B}(v)$ satisfies:

$$(v - Proj_{\Omega,B}(v))^T B(u - Proj_{\Omega,B}(v)) \leq 0, \quad \forall \quad u \in \Omega. \tag{2.5}$$

Definition 2.2.2 *The mapping $s(v) : R^m \to R^m$ is Lipschitz continuous on Ω, if there is a constant L_s such that*

$$\|s(v') - s(v'')\|_2 \leq L_s \|v' - v''\|_2, \quad \forall \quad v', v'' \in \Omega. \tag{2.6}$$

Definition 2.2.3 (Ortega and Rheinboldt, 1970) $s(v) : R^m \to R^m$ is

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(*a*) *monotone on* Ω *if*

$$[s(v') - s(v'')]^T(v' - v'') \geq 0 \quad \forall \quad v', v'' \in \Omega; \tag{2.7}$$

(*b*) *pseudo-monotone on* Ω *if*

$s(v')^T(v'' - v') \geq 0$ *implies*

$$s(v'')^T(v'' - v') \geq 0, \quad \forall \quad v', v'' \in \Omega; \tag{2.8}$$

(*c*) *strictly monotone on* Ω *if*

$$[s(v') - s(v'')]^T(v' - v'') > 0, \quad \forall \quad v', v'' \in \Omega, \quad v' \neq v''; \tag{2.9}$$

(*d*)*strongly monotone on* Ω *if there exists an* $\alpha > 0$ *such that*

$$[s(v') - s(v'')]^T(v' - v'') \geq \alpha ||v' - v''||_2^2, \quad \forall \quad v', v'' \in \Omega; \tag{2.10}$$

(*e*) *coercive with respect to* Ω *if there ezits a vector* $v^0 \in \Omega$ *such that*

$$\lim_{v \in \Omega, ||v||_2 \to \infty} \frac{s(v)^T(v - v^0)}{||v||} = \infty \tag{2.11}$$

where $||\cdot||_2$ *denotes a Euclidean norm in* R^m.

When $s(v)$ is the gradient of some differentiable function f, then the various concepts of monotonicity previously defined correspond, respectively, to convexity, pseudo convexity, strict convexity, and strong convexity of f on Ω. For a differentiable function, we have the following characterizations.

Definition 2.2.4 (Auslender, 1976) *The mapping* $s(v) : R^m \to R^m$ *is*

(*a*) *monotone on* Ω *if*

$$(v' - v'')^T \nabla s(v'')(v' - v'') \geq 0, \quad \forall \quad v', v'' \in \Omega; \tag{2.12}$$

(*b*) *strongly monotone if*

$$(v' - v'')^T \nabla s(v'')(v' - v'') \geq \lambda ||v' - v''||_2^2, \quad \forall \quad v', v'' \in \Omega \tag{2.13}$$

for some positive number λ.

CHAPTER 2. PROBLEM STATEMENT AND PRELIMINARIES

Now we consider some concepts of nondifferentiable functions which are important in the development of our general descent framework.

Definition 2.2.5 (Clarke, 1975) *The generalized gradient of $f(v)$ at v, denoted $\partial f(v)$, is the convex hull of the set of limits of the form* $\lim \nabla f(v + \Delta v_i)$ *where* $\Delta v_i \to 0$ *as* $i \to \infty$.

It follows that $\partial f(v)$ is a nonempty convex compact set (Rockafellar, 1970, Theorem 17.2). The mapping $\partial f(v)$ from R^m to the subset of R^m is also seen to be upper semicontinuous (u.s.c.): if v_i and u_i are sequences tending to v and u respectively, and if v_i belongs to $\partial f(u_i)$ for each i, then v belongs to $\partial f(u)$.

The vector ρ is a subgradient of f at v if, for all $v \in R^m$,

$$f(v + \Delta v) - f(v) \geq \Delta v \rho$$

where the right hand side denotes the usual inner product.

Proposition 2.2.1 (Rockafellar, 1970) *If f is convex, then $\partial f(v)$ is the set of subgradients of f at v.*

Definition 2.2.6 *$f^{\circ}(v;d)$ denotes the generalized directional derivative of f at v along the direction d,*

$$f^{\circ}(v;d) = \lim_{\Delta v \to 0,\ \delta \to 0} \sup [f(v + \Delta v + \delta d) - f(v + \Delta v)]/\delta.$$

Proposition 2.2.2 (Clarke, 1975) $f^{\circ}(v;d) = \sup_{\zeta \in \partial f(v)} \{\zeta d\}$.

Proposition 2.2.3 (Clarke, 1975) *The following are equivalent:*

(a) $\partial f(v) = \{\zeta\}$, a singleton;
(b) $\nabla f(v)$ exists, $\nabla f(v) = \zeta$, and $\nabla f(v)$ is continuous at v relative to the set upon which it exists.

Theorem 2.2.1 *Let U be a sequentially compact space, and let $g : R^m \times U \to R$ have the following properties:*
(a) $g(v,u)$ is u.s.c. in (v,u);
(b) g is locally Lipschitz continuous in v, uniformly for u in U;
(c) $\nabla_v g(v,u)$ is the derivative with respect to v;

(*d*) $\partial_v g(v,u)$ *is u.s.c. in* (v,u).

If we let

$$f(v) = \max\{g(v,u) : u \in U\},$$

then we have

(1) f *is locally Lipschitz;*
(2) $f'(v;d) = \max\{\zeta d \mid \zeta \in \partial_v g(v,u), u \in H(v)\}$, *where*

$$H(v) = \{u \in U : g(v,u) = f(v)\}.$$

(3) $\partial f(v)$ *is the convex hull of* $\{\partial_v g(v,u) : u \in H(v)\}$, *i.e.,*

$$\partial f(v) = \mathrm{Co}\{\partial_v g(v,u) : u \in H(v)\}.$$

The conditions on g also hold when $\nabla_v g(v,u)$ exists and is continuous in (v,u), and we thereby obtain a generalization of a theorem originally due to Danskin (1966).

To introduce some of the basic results in algebra which will be used in some proofs of convergence theorems, we give the following definitions, notations and propositions.

As mentioned above, if B is a symmetric positive definite matrix, the B-norm of a vector x is given by

$$\|x\|_B = (x^T B x)^{1/2}. \tag{2.14}$$

If B is the identity matrix, then the B-norm becomes the standard Euclidean noted by $\|\cdot\|_2$. The induced B-norm of a matrix G is

$$\|G\|_B = \max\{\|Gx\|_B \ : \ \|x\|_B = 1\}. \tag{2.15}$$

The Frobenius norm of a matrix G (not necessarily square) is

$$\|G\|_F = (\sum_i \|g_i\|_2^2)^{1/2} \tag{2.16}$$

where g_i is the ith column vector of G, i.e., $G = (..., g_i, ...)$. It is known (Lancaster and Tismenetsky, 1985) that if G^+ and G^* are the Moore-Penrose inverse and the conjugate of any matrix G, then we have

$$\begin{aligned} GG^+G &= G \\ G^+GG^+ &= G^+ \\ (GG^+)^* &= GG^+ \\ (G^+G)^* &= G^+G. \end{aligned} \tag{2.17}$$

It is also known that for a square matrix G and an nonzero vector x we have

$$x^TGx \geq \lambda_{min}||x||_2^2 \tag{2.18}$$

$$x^TGx \leq \lambda_{max}||x||_2^2 \tag{2.19}$$

where λ_{min} and λ_{max} are respectively the minimum and maximum eigenvalues of the symmetric part of the matrix G and they are real numbers.

We are given two nonzero vectors x and y and a matrix G where $G \in R^{n \times m}$ is assumed to be one-to-one and onto, i.e., G is Moore-Penrose invertible. Now we consider the following transformation:

$$x = Gy \tag{2.20}$$

The results of the following three propositions about the linear transformation (2.20) are used in the proof of convergence of the algorithms.

Proposition 2.2.4

1) $||x||_2 \leq ||G||_F \cdot ||y||_2$. (2.21)

2) $||x||_2 \geq ||G^+||_F^{-1} \cdot ||y||_2$. (2.22)

Proof.

1) By definition of norm consistency, we have

$$||x||_2 = ||Gy||_2 \leq ||G||_F \cdot ||y||_2.$$

2) Since G^+ is the Moore-Penrose inverse of G, we have $y = G^+x$ and

$$||y||_2 = ||G^+x||_2 \leq ||G^+||_F \cdot ||x||_2$$

$$||x||_2 \geq ||G^+||_F^{-1} \cdot ||y||_2. \qquad \square$$

Proposition 2.2.5

$$1) \quad ||x||_2 \leq \lambda_{min}^{-1/2}||x||_B \tag{2.23}$$

$$2) \quad ||x||_2 \geq \lambda_{max}^{-1/2}||x||_B \tag{2.24}$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues of the symmetric and positive definite matrix B.

Proof.

By definition, we have

$$||x||_2 = (\sum_i \alpha_i^2)^{1/2} \text{ and } ||x||_B = (x \cdot Bx)^{1/2} = (\sum_i \alpha_i^2 \lambda_i)^{1/2}$$

where α_i is the ith principal component of x and λ_i is the ith eigenvalue of B. Since $\lambda_{min} > 0$, we have

$$||x||_B \geq \lambda_{min}^{1/2}(\sum_i \alpha_i^2)^{1/2} = \lambda_{min}^{1/2}||x||_2 \leftrightarrow ||x||_2 \leq \lambda_{min}^{-1/2}||x||_B$$

$$||x||_B \leq \lambda_{max}^{1/2}(\sum_i \alpha_i^2)^{1/2} = \lambda_{max}^{1/2}||x||_2 \leftrightarrow ||x||_2 \geq \lambda_{max}^{-1/2}||x||_B. \qquad \square$$

Proposition 2.2.6

1) $\|x\|_2^2 \leq \beta \|y\|_B^2$ (2.25)

2) $\|x\|_2^2 \geq \gamma \|y\|_B^2$ (2.26)

where B is a symmetric and positive definite matrix, $\beta = \|G\|_F^2 \lambda_{min}^{-1}$ and $\gamma = \|G^+\|_F^{-2} \lambda_{max}^{-1}$.

Proof.

1) $\|x\|_2^2 \leq \|G\|_F^2 \cdot \|y\|_2^2 \leq \|G\|_F^2 \lambda_{min}^{-1} \|y\|_B^2 = \beta \|y\|_B^2$.

2) $\|x\|_2^2 \geq \|G^+\|_F^{-2} \cdot \|y\|_2^2 \geq \|G^+\|_F^{-2} \lambda_{max}^{-1} \|y\|_B^2 = \gamma \|y\|_B^2$. □

Chapter 3

Literature Review

In this chapter, we consider in some detail the state of the art of results related to the VIP where our interest is on the finite VIP and the theoretical analysis (solution characterizations, existence, uniqueness), various reformulations of VIP. and the corresponding solution approaches. We mention that since Harker and Pang (1990) provided an excellent review on this topic recently we list some of the basic results of their survey for completeness and present what they did not cover at the same time. However, this survey is basically different from them in that

- it emphasizes on the reformulations of the VIP and the solution approaches;
- its presentation of the solution approaches is based on three ideas of constructing algorithms (This is a personal view. It is interesting to extend the view, which helps to develop some new algorithms);
- new developments in the optimization approaches to solve the VIP are included;
- several results in Europe which are apparently missing are discussed such as the Migdalas' and Cohen's work.

There are some similarities in the following senses.

- The most results in the VIP analysis here came from their survey, since the research in the existence and uniqueness of the VIP is not our main topic.

3.1 VIP analysis

After the VIP is stated, it is natural to ask following questions. Does the solutions of the VIP exist? If they exist, whether they are unique. This question occur since, as long as we formulate a VIP model, we want to see a solution of the model. And also from both computational and practical point of view, we want see if we have the solutions and the solutions are unique. Extensive research has been done during the past two decades on the existence and uniqueness of solutions to the VIP. The results are so numerous that it is impossible to discuss all of them. In this section, we have chosen to provide the most basic results and some of their consequences; other results can be found in the references.

The most basic assumption on the existence of a solution to the $VIP(\Omega, s)$ requires the set Ω to be compact and convex, and the mapping s to be continuous. Based on this basic result, many extended corollaries can be derived by replacing the compactness of Ω with additional conditions on $s(v)$. We define $Proj_{B,\Omega}(v)$ as a projection of v onto Ω with respect to the norm B. Applying Brouwer's fixed-point theorem to the mapping

$$H(v) = Proj_{B,\Omega}(v - \alpha B^{-1}s(v)),$$

one can prove the following theorem on the basic existence result.

Theorem 3.1.1 (Eaves [61], 1971) *Let $H : \Omega \subset R^m \to R^m$ be continuous on the compact and convex set Ω and suppose that $H(\Omega) \subset \Omega$. Then H has a fixed point in Ω.*

Theorem 3.1.2 (Eaves [61], 1971, Hartman and Stampacchia,1966) *Let Ω be a nonempty, compact and convex subset of R^m and let s be a continuous mapping from Ω into R^m. Then there exists a solution to the problem $VIP(\Omega, s)$.*

It is easy to prove that the solution set of the VIP is convex if it is nonempty. In general, the VIP can have more than one solution. For the uniqueness of the solution of the VIP, we have

Proposition 3.1.1 *If s is strictly monotone on Ω, then the VIP has at most one solution.*

It is noted that the Proposition 3.1.1 does not guarantees the existence of a solution to the VIP. In general the coercivity property of s is used to establish existence.

Theorem 3.1.3 (Hartman and Stampacchia, 1966) *Let Ω be a nonempty, closed, convex subset of R^m and let s be a continuous mapping from Ω into R^m. If s is coercive with respect to Ω, then the VIP has a nonempty, compact solution set.*

Since strong monotonicity implies both coercivity and strict monotonity we can, by combining Theorem 3.1.3 and Proposition 3.1.1, derive the following existence and uniqueness result for the VIP with the strongly monotone function $s(v)$.

Corollary 3.1.1 *Let Ω be a nonempty, closed, convex subset of R^m and let s be a continuous mapping from Ω into R^m. If s is strongly monotone with respect to Ω, then there exists a unique solution to the VIP.*

3.2 Various formulations of VIP

In this section, we look at the various formulations of VIP. This is interesting because a different formulation, in general, requires a different solution approach, which is our concern. These formulations are: the nonlinear complementarity formulation, the gap-function type formulation, Smith's formulation, Fukushima's formulation, Pang's formulation and the fixed point formulation.

3.2.1 Nonlinear complementarity

The nonlinear complementarity problem $NCP(s)$ is a special case of the VIP. We have the following definition.

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Definition 3.2.1 *Given a vector s being a mapping from R^m into itself and $v \in R^m$, the nonlinear complementarity problem, denoted by $NCP(s)$, is to find a vector $v^* \in R^m$ such that*

$$v^* \geq 0, \quad s(v^*) \in R^m \text{ and } s(v^*)^T v^* = 0. \tag{3.1}$$

We define the linear complementarity problem which is denoted by $LCP(q, M)$ as the Definition (3.2.1) where $s(v) = q + Mv$ for some given vector $q \in R^m$ and matrix $M \in R^m \times R^m$.

The generalized complementarity problem is defined by replacing R^m with an arbitrary convex cone.

Definition 3.2.2 *Given Ω being a convex cone in R^m and s being a mapping from R^m into itself, the generalized complementarity problem, denoted by $GCP(\Omega, s)$, is to find a vector $v^* \in \Omega$ such that*

$$s(v^*) \in \Omega^* \text{ and } s(v^*)^T v^* = 0, \tag{3.2}$$

where Ω^* denotes the dual cone of Ω, i.e.,

$$\Omega^* = \{u \in R^m : u^T v \geq 0, \ \forall \ v \in \Omega\}.$$

We denote that the relationship between the generalized complementarity problem and the variational inequality problem was established first by Karamardian (1971).

Proposition 3.2.1 (Karamardian, 1971) *Suppose that Ω is a convex cone. Then $v^* \in \Omega$ solves the problem $VIP(\Omega, s)$ if and only if v^* solves the $GCP(\Omega, s)$.*

There are several optimization formulations available for the VIP. In addition to the optimization formulation mentioned above, where the symmetry principle is satisfied, we have the following results where the symmetric condition on the s function is not required.

3.2.2 Gap-function type formulation

Definition 3.2.3 *The gap function $g(v)$ for the problem $VIP(\Omega, s)$ is given by*

$$g(v) = \max_{u \in \Omega} s(v)^T (v - u). \tag{3.3}$$

Thus, we have the following equivalent mathematical programming formulation of any VIP.

Proposition 3.2.2 (Hearn, 1982) *The vector $v^* \in \Omega$ is a solution to the $VIP(\Omega, s)$ if and only if v^* solves*

$$\min_{v \in \Omega} g(v) = \min_{v \in \Omega} \max_{u \in \Omega} s(v)^T (v - u) \tag{3.4}$$

and $g(v^*) = 0$.

While, in general, like most minimax problems of interest, the gap function is nondifferentiable and nonconvex, we mention that if $s(v)$ is monotone, any stationary point v^* of $g(v)$, i.e., $g'(v^*; d) \geq 0$, is a solution of VIP (Marcotte, 1985) which is a minimal solution ($f(v^*) \leq f(v)$). This property is interesting, since we can apply any local optimal method to find a local solution which is, in fact, a global solution. In particular, a globally convergent algorithm using the gap function as a merit function has been proposed by Marcotte (1985).

3.2.3 Smith's formulation

Definition 3.2.4 *Let $\Phi(\beta) = \max\{0, \beta\}$ and $\Phi^\rho(\beta) = (\max\{0, \beta\})^\rho$, where $\rho \geq 1$ is integer and let Ω be the convex hull of $\{u_1, u_2, ..., u_r\}$. Then*

$$V_\rho(v) = \sum_{i=1}^{r} \Phi^\rho(s(v)^T (v - u_i)), \quad \forall \ v \in \Omega. \tag{3.5}$$

Theorem 3.2.1 (Smith, 1983) *Assume for any integer $\rho \geq 0$.*

1. *$V_\rho(v) > 0, \forall v \in \Omega$ and $v \notin \Omega^*$ where Ω^* is the solution set;*
2. *$V_\rho(v) = 0, \forall v \in \Omega^*$;*

3. $V_\rho(v^*) = \min_{v \in \Omega} V_\rho(v)$.

Note that letting $V_\infty(v) = \lim_{\rho \to \infty}(V_\rho(v))^{\frac{1}{\rho}}, \forall \ v \in \Omega^*$, we have $g(v) = V_\infty(v)$. That means that Smith's class of functions includes the gap function as a limiting case. For $2 \leq \rho < \infty, V_\rho(v)$ is differentiable everywhere and its gradient is

$$\nabla V_\rho = \sum_{i=1}^{r} \rho \Phi^{\rho-1}(s(v)^T(v-u_i))(s(v) + \nabla s(v)^T(v-u_i)).$$

3.2.4 Convexity discussion

If certain conditions are satisfied, the $g(v)$ and $V_\rho(v)$ are convex functions.

Theorem 3.2.2 (Hearn, 1982) *Let $s(v)^T v$ be convex and assume that each component of $s(v)$ is concave in Ω. Then $g(v)$ is convex in Ω.*

Theorem 3.2.3 (Hearn, 1982) *If $g(v)$ is convex, then*

$$\partial g(v) = \ \mathrm{Co}\{s(v) + \nabla s(v)(v-u) : u \in H(v)\}$$

where $\partial g(v)$ denotes the subgradients of $g(v)$ at v, $\nabla s(v)$ is the Jacobian of $s(v)$ and $H(v)$ is the set of solution to the problem

$$\max_{u \in \Omega}\{s(v)^T(v-u)\}.$$

Theorem 3.2.4 (Hearn, 1982) *$g(v)$ is differentiable at v if and only if $H(v)$ is a singleton.*

Nguyen and Dupuis (1984) introduced the following definition.

Definition 3.2.5 (Nguyen and Dupuis, 1984)

$$G(v) = \min_{u \in \Omega}\{s(u)^T(u-v)\}, \quad \forall v \in \Omega.$$

Hearn et al (1984) observed that $G(v)$ is also concave and establish the relationship between $g(v)$ and $G(v)$ as follows:

Theorem 3.2.5 (Hearn, Lawphongpanich and Nguyen, 1984) *$G(v)$ is a concave function on Ω.*

Theorem 3.2.6 *$\max G(v)$ and $\min g(v)$ are dual in the sense that*

$$g(v') \geq 0 \geq G(v''), \quad \forall \; v', v'' \in \Omega,$$

$$\min_{v \in \Omega} g(v) = 0 = \max_{v \in \Omega} G(v).$$

Theorem 3.2.7 $\partial G(v) = \text{Co}\{-s(u) : u \in H(v)\}$

where $H(v)$ is the set of solutions to $\min\{s(u)^T(u-v)\}$. The $G(v)$ is differentiable at v if and only if $H(v)$ is a singleton.

While the development by Smith does not address the convexity of $V_\rho(v)$, Hearn, Lawphongpanich and Nguyen (1984) investigate this problem.

Theorem 3.2.8 (Hearn, Lawphongpanich and Nguyen, 1984)
If $\Phi(s(v)^T(v-u_i)\sum_j^M [v-u_i]_j)H_j(v)$ is positive semidefinite for all $i = 1, ...r$, for all $v \in \Omega^*$ and if $s(v)$ is monotone, then $V_\rho(v)$ is convex for $\rho \geq 3$, where $H_j(v)$ is the Hessian matrix for the function

$$s_j(v), \quad j = 1, ..., M.$$

Theorem 3.2.9 (Hearn, Lawphongpanich and Nguyen, 1984)
Let $Z_i = s(v)^T(v-u_i)$ for $i = 1, ..., r$. If $\nabla^2 Z_i(v)$ is positive semidefinite for all $v \in \Omega^*$, then $V_\rho(v)$ is convex for all integers $\rho \geq 1$ and for $\rho = \infty$.

3.2.5 Fukushima's formulation

As we see, Smith's formulation requires that Ω be the convex hull of a finite set of points $\{u_1, ..., u_r\}$. In some cases, this is not true. A significant contribution to the reformulation of VIP as an equivalent optimization problem is that of Fukushima (1989) who developed a continuously differentiable optimization formulation for VIP when $\nabla s(v)$ is asymmetric.

Definition 3.2.6

$$\textit{Let} \quad f(v) = -s(v)^T(H(v)-v) - \frac{1}{2}(H(v)-v)^T B(H(v)-v) \quad (3.6)$$

where $H(v) = Proj_{B,\Omega}(v - B^{-1}s(v))$ *and* B *is a symmetric positive definite matrix.*

Consider the optimization problem

$$\min\{f(v) : v \in \Omega\}. \quad (3.7)$$

Theorem 3.2.10 (Fukushima, 1989) *Let* f *be defined by (3.6). Then* $f(v) \geq 0$ *for all* $v \in \Omega$, *and* $f(v^*) = 0$ *if and only if* v^* *solves the* VIP. *Hence* v^* *solves the optimization problem (3.7) if and only if* v^* *solves the* VIP.

Theorem 3.2.11 (Fukushima, 1989) *If* $s(v)$ *is continuous on* Ω, *then the function* $f(v)$ *is also continuous. Furthermore, if* $s(v)$ *is continuously differentiable, then* f *is also continuously differentiable and its gradient is given by*

$$\nabla f(v) = s(v) - [\nabla s(v) - B](H(v) - v).$$

3.2.6 Pang's formulation

Pang(1990) uses a theory of B-differentiable system of nonlinear equations to solve the VIP.

Definition 3.2.7 *A function* $H(v) : R^m \to R^n$ *is said to be* B*-differentiable at a point* v *if there exists a function* $BH(v,z)$, *called the* B*-derivative of* H *at* v, *which is positively homogeneous of degree 1 (i.e.,* $BH(v,\lambda z) = \lambda BH(v,z)$ *for all* $z \in R^m$ *and all* $\lambda \geq 0$*), such that*

$$\lim_{z \to 0} \frac{H(v+z) - H(v) - BH(v,z)}{||z||} = 0.$$

If H *is* B*-differentiable for all points in the set* Ω, *then* H *is said to be* B*-differentiable in* Ω.

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The one fundamental distinction between a B-differentiable function and a F-differentiable function is the absence of linearity in $BH(v,z)$.

It is important to mention that $BH(v,z)$ is, in fact, F-differential in the sense of Ortega and Rheinboldt (1970, p66) and the generalized directional derivative (Pang, 1990).

There are two ways to convert VIP into a system of B-differentiable equations. The first way is through a mixed nonlinear complementarity problem and is applicable when the set Ω is defined in terms of differentiable inequalities and equations:

$$\Omega = \{v \in R^m : g_i(v) \leq 0, \quad i = 1,...,n; \quad h_j(v) = 0, \quad j = 1,..,l\}$$

where g_i and $h_j : R^m \to R$ are all twice continuously differentiable. If v is a solution of VIP and if some constraint qualification holds at v, then there exist multipliers $\{\lambda_i : i = 1,...n\}$ and $\{u_j : j = 1,...,l\}$ such that the following conditions hold:

$$0 = s(v) + \sum_{i=1}^{n} \lambda_i \nabla g_i(v) + \sum_{j=1}^{l} u_j \nabla h_j(v), \tag{3.8}$$

$$\lambda_i \geq 0, \quad \lambda_i g_i(v) = 0, \qquad i = 1,..,n. \tag{3.9}$$

Let $g : R^m \to R^n$ and $h : R^m \to R^l$ denote the vector-valued function of the g_i and h_j respectively. By defining the function $H : R^m \times R^n \times R^l \to R^m \times R^n \times R^l$ as

$$H\begin{pmatrix} v \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} s(v) + \sum_{i=1}^{n} \lambda_i \nabla g_i(v) + \sum_{j=1}^{l} \mu_j \nabla h_j(v) \\ \min(\lambda, -g(v)) \\ h(v) \end{pmatrix}. \tag{3.10}$$

It is easy to see that the equation $H(v,\lambda,\mu) = 0$ summarizes the conditions (3.8) and (3.9). Consequently, finding a zero of this H function would solve the problem VIP. The function H is obviously B-differentiable. It should be pointed out that the function H will not be F-differentiable at a point unless strict complementarity holds there, i.e., $\lambda_i \neq -g_i(v), i = 1,2,...,n$.

Another approach to converting the VIP into a system of B-differentiable equations is based on the the fact that a vector v solves VIP if and only if it is a zero of the function $H : R^m \rightarrow R^m$ defined by

$$H(v) = v - Proj_\Omega(v - s(v))$$

where $Proj_\Omega(z)$ denotes the projection of the point z onto the set Ω. H is then B-differentiable.

Proposition 3.2.3 *Let $H : R^m \rightarrow R^m$ be B-differentiable and let $g : R^m \rightarrow R$ be defined by*

$$g(v) = \frac{1}{2} H(v)^T H(v) \tag{3.11}$$

where $v \in \Omega$. Then $g(v^*) = 0$, if and only if v^* is the solution of VIP.

3.2.7 Optimization formulation discussion

It is interesting to look at the advantages and disadvantages of each formulation. The gap function is computed by solving a linear program, in particular, finding minimum path trees if applied in the network equilibrium case but it is not differentiable for all values of v. It is useful as a monitor of simplicial decomposition (column generating) schemes, especially as a stopping criteria and to guide the dropping of columns as in Lawphongpanich and Hearn (1982). The dual gap function is always concave for all feasible flows v, but it may be difficult to evaluate. Smith's family of functions, for $2 \leq \rho < \infty$, is differentiable and hence potentially very useful when the extreme points $\{u_1, u_2, ..., u_r\}$ are known, as in simplicial decomposition. The Fukushima's formulation is differentiable, but is difficult to evaluate unless Ω has a special structure which can be possible by using the outer-approximation method (Fukushima, 1986). This function has the very nice property that any local optimum is a global optimum under some usual conditions. Since the evaluation of the function requires a projection, if the projection can be easily carried out, then the method is very promising. Pang's formulation has the same properties as Fukushima's one except the B-differentiability instead of differentiability of the function which could be also very promising.

3.2.8 The fixed point formulation

Proposition 3.2.4 (Eaves [61], 1971) *Let Ω be a nonempty closed convex subset of R^m and let B be any $m \times m$ symmetric positive definite matrix. Then v^* solves the problem $VIP(\Omega, s)$ if and only if*

$$v^* = Proj_{B,\Omega}(v^* - B^{-1}s(v^*));$$

i.e., if and only if v^* is a fixed point of the mapping $H : R^m \to R^m$ defined by

$$H(v) = Proj_{B,\Omega}(v - B^{-1}s(v)). \tag{3.12}$$

This formulation plays an important role in proving the basic existence results by applying Brouwer's fixed point theorem.

3.3 Solution approaches

As discussed before, there is a relative short history for the development of algorithms for solving the VIP. If $\nabla s(v)$ is symmetric and positive definite, the VIP is known to be equivalent to a simple convex mathematical program. One of the solution methods of the problem is the Frank and Wolfe method first proposed by LeBlanc et al (1974) for the network equilibrium problem. Our discussion focuses on the VIP where the $\nabla s(v)$ is asymmetric.

From the literature, we find that there are three basic ideas available to develop the solution methods for the VIP. The first idea is to establish an approximation of the cost function $s(v)$. More specificly, we have the linear and nonlinear approximation of $s(v)$ and some perturbation on $s(v)$. The approaches developed on the idea are called the function-related algorithms, which are the iterative methods and proximal point algorithms. The second idea is to redefine the feasible region Ω at each iteration so that the subproblem can be solved easily. The methods are referred to the region-related algorithms. Among these methods, we have the simplicial decomposition (inner approximation) methods and the relaxed projection (outer approximation) methods. The third idea is to reformulate the VIP to form some equivalent optimization problems. These are the optimization approaches.

In addition to the methods based on purely one of these ideas, there are in fact some developed methods for the VIP which employ a mixture of these ideas. For example, the Bertsekas and Gafni's method (1982) is the projection method (a kind of linear approximation method) on the restricted region (simplicial decomposition). Pang and Yu (1984) also used the linear approximation method and the simplicial decomposition method to derive their method. How to relate the first idea to the third idea is an interesting research field which is the topic of the current research. And it seems that there is no much work done based on the second idea and third idea, which is a future research. We now discuss each of these methods in turn.

3.3.1 Function-related algorithms

In this family, there are two kinds of algorithms: iterative algorithms (framework) and proximal point algorithms. A general iterative framework for solving the $VIP(\Omega, s)$ was established in detail by Pang and Chan (1982) who gave a general treatment of convergence properties of various algorithms. It works as follows. A sequence $\{v^l\} \subset \Omega$ is generated such that each v^{l+1} solves variational inequality subproblem $VIP(\Omega, s^l)$, i.e.,

$$s^l(v^{l+1})^T(v - v^{l+1}) \geq 0, \quad \forall \ v \in \Omega$$

where $s^l(v)$ is some approximation to $s(v)$ at v^l. The algorithm continues until a suitable criterion is satisfied. There are two basic choices for this approximation, i.e., $s^l(v)$ is either a linear or nonlinear function. For the linear approximation:

$$s^l(v) = s(v^l) + B(v^l)(v - v^l)$$

where $B(v^l)$ is an $m \times m$ positive definite matrix. We have the following several methods depending on $B(v^l)$:

$$B(v^l) = \begin{cases} = \nabla s(v^l) & \text{(Newton)} \\ \approx \nabla s(v^l) & \text{(Quasi} - \text{Newton)} \\ = D(v^l) & \text{(Linearized Jacobi)} \\ = \frac{UL(v^l)+D(v^l)}{w} & \text{(Successive overrelaxation)} \\ = \frac{1}{2}[\nabla s(v^l) + \nabla s(v^l)^T] & \text{(Symmetrized Newton)} \\ = \overline{B} & \text{(Projection)} \\ = B' & \text{(Asymmetric projection)} \end{cases}$$

where $D(v^l)$ is the diagonal part of $\nabla s(v^l)$, $UL(v^l)$ is the upper or lower triangular part of $\nabla s(v^l)$, w^* is a scalar parameter $\in (0,2)$ and $\overline{B}$ is a fixed symmetric positive definite matrix and B^l is a positive definite matrix.

Now we discuss these methods. Newton's method is known to have a quadratic convergence rate. However, as in the case of solving a system of nonlinear equations and an optimization problem, there are several well-recognized drawbacks with this method:

1. we need to compute the $m \times m$ Jacobian matrix $\nabla s(v^l)$ at each iteration,
2. and the solution of each variational inequality subproblem $VIP(\Omega, s^l)$ is more difficult.

Many attempts have been done to overcome these difficulties. The first drawback was overcome by having Quasi-Newton methods. The second was considered by employing some other linear approximation methods at the expense of linear (or super linear) convergence instead of a quadratic convergence.

We note that the linear approximation methods which make use of a symmetric matrix $B(v^l)$ at each iteration are the linearized Jacobi, symmetrized Newton (Hammond, 1984) and projection algorithms (Dafermos, 1980, 1983, Bertsekas and Gafni, 1982). One important characteristic of these methods are that each subproblem is in fact equivalent to a convex quadratic problem for which there are many well-known solution procedures available depending on the structure of the subproblem. In general, it is relatively easy to find a solution of such subproblem. On the other hand, these methods have two problems. Firstly, they require strong conditions on $s(v)$ for the global convergence and secondly, the convergence rate is linear or superlinear, which implies a slow convergence as compared with the Newton methods.

Among all these methods, the most simple method is the projection method which takes $B(v^l)$ to be a fixed symmetric, positive definite matrix B for all l. The method has received the most attention

of the aforementioned methods. The advantage of the method is that it is globally convergent. However, it is only applicable to the problem $VIP(\Omega, s)$ when s is strongly monotone. In essence, a projection method which is just a simple fixed point iteration of the mapping H defined by (3.12) was one of the first algorithms used on finite dimensional problems (Dafermos, 1980). It is noted that the method has a slow convergence in numerical computations since for all iterations $B(v^l)$ is fixed and does not use the information of $\nabla s(v^l)$. In order to take advantage of the information of $\nabla s(v^l)$ and keep about the same procedure to compute, the linearized Jacobi method was developed by taking $B(v^l)$ to be a diagonal matrix of $\nabla s(v^l)$ at each iteration and leads to an optimization problem with a separable quadratic objective function. In order to use the full information on $\nabla s(v)$, Hammond and Magnanti (1987) developed the symmetrized Newton method, also called the contracting ellipsoid method. The method is a simple variant of the general linearization scheme in which the symmetric part of $\nabla s(v)$ is used to define the linear approximation.

Dafermos (1983) and Pang and Chan (1982) have shown that if B' and s satisfy a certain contraction condition, then the asymmetric projection method converges to a solution of $VIP(\Omega, s)$. Unfortunately, this condition implies that s is strictly monotone. Tseng (1990) broadened the condition so that it is applicable to all monotone (not necessarily strictly monotone) affine variational inequality problems by a forward-backward matrix splitting algorithm.

Along the same idea in the solution of linear equations, besides the symmetric methods, there are other linear approximation schemes. The most notable is the linearized successive overrelaxation (SOR) method described above. In application, this method is computationally faster in solving the nonlinear complementarity problem because, in this case, each resulting linear complementarity subproblem is defined by a triangular matrix M and is thus easily solvable. The drawback of the SOR methods is that they require some estimation of optimal w^* in order for a faster convergence. For further information on this algorithm, the reader is referred to Ahn (1982), Ortega and Rheinboldt (1970) and Pang (1981).

Another algorithm, not discussed so far (Migdalas, 1988), is the successive approximation algorithm where $s^l(v)$ is given by $s_i^l(v) = s_i(v^l)$. Thus each subproblem is in fact a linear program, in particular, an all-or-nothing assignment problem in the network equilibrium problem, which is considerably easier to solve.

Now we look at nonlinear approximation schemes for solving VIP. The most popular is the nonlinear Jacobi (also called relaxation or diagonalization) algorithm (Ortega and Rheinboldt, 1970, Ahn, 1979, Pang and Chan, 1982, Friesz, Harker and Tobin, 1984, Florian and Spiess, 1982). It is interesting to note that the method in the PIES model actually reduces to that algorithm. The basic idea of this algorithm is to fix the cross effects of v on $s(v)$, i.e., produce a separable nonlinear map at each iteration to form an equivalent optimization problem (called the VIP subproblem) and solve the VIP subproblem by some conventional methods to obtain a new iterate instead of using a linear approximation for $s^l(v)$. Due to the separability of s^l, each subproblem VIP can again be converted into the form of an optimization problem; this is why this method was so successful in applications. However, it is still expensive to solve the optimization problem.

Between the nonlinear Jacobi and the successive approximation algorithms exists a streamlined algorithm (see Migdalas, 1988). It takes advantages of the better-approximation benefits of the former and the computational simplicity of the latter. In this method, the subproblem of the former method is approximately solved by performing only one Frank-Wolfe iteration and a line search. Computed results are not conclusive.

All the methods discussed above, with the exception of the projection method, are shown to be locally convergent.

Cohen (1988) extended his so-called "auxiliary problem principle" to develop a general class of iterative computational algorithms which is even more general than the linear approximation method.

Numerical experience has indicated (Friesz, 1985) that the linear approximation methods, when used in the network equilibrium problem, tend to converge anyway to a solution (not necessarily unique) not under the convergence conditions discussed. This suggests that those sufficient conditions may be weakened for local convergence..

The proximal point algorithm is another algorithm for the VIP. For the VIP, the algorithm works as follows. Given v^l, let v^{l+1} solve the variational inequality problem $VIP(\Omega, s + \epsilon_l I)$ where ϵ_l is an arbitrary positive scalar and I is the identity map. It is important to note that if s is monotone, then the mapping $s^l = s + \epsilon_l I$ is strongly monotone. Thus, each subproblem $VIP(\Omega, s^l)$ has a unique solution. Rockafellar ([258], 1976) has established that if each v^{l+1} is selected according to an appropriate inexact rule, then the sequence v^{l+1} is bounded if and only if the problem VIP has a solution. This algorithm is interesting since it does not require the strong monotonicity of the function $s(v)$, i.e., it works under weaker conditions as compared with other methods in the basic iterative scheme.

3.3.2 Region-related algorithms

These algorithms include the simplicial decomposition methods and the relaxed projection method. The simplicial decomposition method for the VIP, which was developed based on the similar ideas in nonlinear programming and column generation in linear programming, is in fact a kind of technique used in the iterative method where each subproblem is solved on a restricted region of Ω expressed as a convex combination of extreme points, which are stored and updated. These methods have been applied to the traffic assignment problem. Bertsekas and Gafni (1982) developed a restriction method operating in the path flow space without column dropping. Lawphongpanich and Hearn (1984) and Pang and Yu (1984) presented the methods in the arc flow space which requires much less storage space. The numerical results indicate that they can be effective computational techniques for large-scale problems. If Ω is a compact polyhedron, it can be expressed as the convex hull of its extreme points. At the beginning of each iteration, a variational inequality subproblem $VIP(\Omega^l, s)$ is solved (either exactly or approximately), where Ω^l denotes the convex hull of a sub-

set of the extreme points of Ω. By using a merit function such as the gap function and some points such as the extreme points generated by performing, say, a shortest path algorithm in the network equilibrium problem, a new subset Ω^{l+1} is obtained and the process continues. Due to the fact that each set Ω^l is the convex hull of a finite number of explicitly known vectors, the solution of each subproblem is very easy, provided that the number of extreme points is small, since it is equivalent to a one dimensional line search in which an inversion of a matrix of dimension of the number of extreme points is performed. Thus, it is important to control the number of extreme points. Lawphongpanich and Hearn (1984) showed that if the gap function is used as the merit function in the above scheme to control the set Ω^l, and if s^l is taken to be the given mapping s for all iterations l, then the method will converge if s is strongly monotone.

Therefore, as mentioned above, the number of extreme points in Ω^l should be controlled if the simplicial decomposition method is to be used in applications. Based on the observation, Lawphongpanich and Hearn (1986) developed restricted simplicial decomposition method. The idea is to limit the number of extreme points kept in each set Ω^l. This scheme is shown to work not only for nonlinear programs but also for the variational inequality problem (Lawphongpanich and Hearn, 1990).

It is worthy to mention that the simplicial decomposition works in a manner of providing inner approximations Ω^l to the set Ω. On the other hand, Fukushima (1986) proposed the use of an outer approximation method by using the projection method. Since in the projection method each subproblem requires another iterative method, Fukushima proposed the outer approximation to the set Ω to make the subproblem easier to solve. More specifically, the proposed algorithm works on a halfspace containing Ω on every iteration. Thus each projection is readily carried out. The next iteration can be expressed explicitly. The method is proven to be globally convergent under the strong monotone assumption on $s(v)$. It would be interesting however to investigate how this outer approximation idea could be applied to other linear approximation algorithms, such as the basic Newton method and optimization

approaches where the the projection is also involved.

3.3.3 Optimization approaches

The algorithms in this category are always related to the optimization reformulations of the VIP discussed in Section 3.2 (see Hearn (1982), Hearn, Lawphongpanich and Nguyen (1984). While Nguyen and Dupuis (1984) proposed a cutting plane algorithm for minimizing the gap function, Marcotte (1985) developed a descent algorithm for an equivalent nonconvex and nondifferentiable optimization problem. Marcotte and Dussault (1989) solved an equivalent optimization formulation for the VIP by solving a sequence of linear programs. Smith (1983, 1984) developed a differentiable function formulation which is solved by a descent algorithm. Fukushima (1989) developed a differentiable formulation and solved it using a descent algorithm. Pang (1990) used Newton's method with line search to solve his optimization formulation (3.11) which, in fact, is a kind of descent direction method applied to a reduced variational inequality at each iteration. Pang's method is different from previous optimization methods in that in terms of line search, Pang's methods operates on the projection function, while Marcotte's method operates on the gap function and Fukushima's method operates on a combination of both the gap and the projection function. The solution of the projection is equivalent to a strictly convex quadratic program and that of the gap function requires a linear program.

3.3.4 Inexact methods

Another idea useful for improving upon the iterative methods was proposed recently by Pang (1986, 1987): inexact iterative methods. The central idea is to define each iterate v^{l+1} as an inexact solution of the subproblem $VIP(\Omega, s^l)$; in other words, at each iteration one would only solve the subproblem up to a certain degree of accuracy. The use of an inexact method immediately raises the question of how one measures the accuracy of the approximate solution to a variational

inequality problem. The following measures of inaccuracy are given:

Projection measure :	$\rho(v) = \|v - Proj_{1,\Omega}(v - s(v))\|$;
Gap function measure :	$g(v) = \sup_{u \in \Omega^*}(u - v)^T s(v)$;
A dual measure :	$d(v) = q + Mx + A^T\lambda + c^T\mu$

where $f = q + Mv, \Omega^* = \{v : b - Av, \lambda \geq 0, \lambda^T\mu = 0, d - cv = 0\}$.

3.4 Quadratic programming

As we see, quadratic programming plays a very important role in the solution of the variational inequality problem. We briefly review two recent research work in this area.

Dussault, Ferland and Lemaire (DFL) (1986) proposed an iterative algorithm for solving a convex quadratic program with one equality constraint and bounded variables. At each iteration, a separable convex quadratic program with the same constraint set is solved. Two variants are analyzed: one which uses an exact line search and the other which uses a unit step size. Recently Pardalos and Kovoor (1990) gave an $O(n)$ algorithm using binary search to solve the Kuhn-Tucker system arising from the quadratic program. In addition, Lin and Pang (1987) presented a very good survey on the iterative methods for large convex quadratic programs.

Chapter 4

Theoretical Development for *VIP*

In this chapter, we consider only the monotone *VIP* and present a general equivalent optimization formulation for it. Based on this formulation, a general descent framework is developed for the general equivalent optimization formulation and a proof of its convergence is given. Then we show that the symmetrized Newton method, Quasi-Newton methods, the linearized Jacobi method, the projection method as well as other methods are various realizations of this general algorithmic framework. In addition, we show that a variant of the descent method, where each subproblem is only solved approximately, is globally convergent under certain conditions. The theoretical development parallels that of Migdalas (1990) for the solution of the convex optimization problem by a regularized Frank-Wolfe (1956) algorithm. Although we only consider *VIP* in this chapter, it is important to note that the same results will be extended to VIP_A in Chapter 5.

This chapter is organized as follows. In Section 1, a general equivalent optimization formulation is presented for *VIP* with a full discussion of its properties. Then, a general descent method is developed for this formulation. Section 2 is devoted to an analysis of the relationship with some other existing algorithms, while, in Section 3, we consider a possible way of applying an approximation strategy to the general framework in order to derive an approximate algorithm. Section 4

provides some conclusions.

4.1 A general equivalent optimization formulation

In this section, we give a general equivalent optimization formulation for *VIP*, discuss its properties and present a general descent framework.

Consider the following optimization problem:

$$\min_v f(v) = -s(v)^T(H(v)-v) - \frac{1}{\alpha}\phi(H(v),v) \tag{4.1}$$

$$\text{s.t. } v \in \Omega$$

$$H(v) = \arg\min_u s(v)^T(u-v) + \frac{1}{\alpha}\phi(u,v)$$

$$\text{s.t. } u \in \Omega$$

where $\phi(u,v) : \Omega \times \Omega \subset R^m \to R^1$ has the following properties:

$-$it is continuously differentiable on $\Omega \times \Omega$; (4.2)

$-$it is nonnegative on $\Omega \times \Omega$; (4.3)

$-$it is strictly convex in u on Ω, (4.4)

i.e., its Hessian matrix with respect to

u is symmetric, positive definite on Ω;

$-\phi(u,v) = 0 \Leftrightarrow u = v$; (4.5)

$-$ α is a positive parameter; (4.6)

$-$ Ω is a closed, convex and nonempty set. (4.7)

Note that if $\alpha = \infty$, then $H(v)$ may be a point-to-set mapping.

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The following several possible forms for $\phi(u,v)$ satisfy (4.2)-(4.7):

$$\phi(u,v) = \frac{1}{2}\|u-v\|_2^2 \tag{4.8}$$

$$\phi(u,v) = \frac{1}{2}(u-v)^T B(u-v), \tag{4.9}$$

where B is a symmetric and positive definite matrix,

$$\phi(u,v) = \frac{1}{2}(u-v)^T B(v)(u-v) \tag{4.10}$$

where $B(v)$ is a symmetric and positive definite matrix. Note that $\phi(u,v)$ is a proximity measure between u and v.

Now, consider the following subproblem of (4.1) for a fixed v:

$$\begin{aligned} \min_u \psi(u,v) &= s(v)^T(u-v) + \frac{1}{\alpha}\phi(u,v) \\ \text{s.t.} \quad & u \in \Omega. \end{aligned} \tag{4.11}$$

It is clear that the program (4.11) is strictly convex and hence is well defined for any α such that $0 < \alpha < \infty$. Furthermore, $\psi(u,v)$ is also a proximity measure between u and v. If $\alpha = \infty$, then the solution to the linear program (4.11) may not be unique.

Proposition 4.1.1 *Assume $\alpha > 0$ and let $H(v)$ be any solution of (4.11). Then (4.11) is equivalent to*

$$[s(v) + \frac{1}{\alpha}\nabla_u\phi(H(v),v)]^T(u-H(v)) \geq 0 \quad \text{for all} \quad u \in \Omega. \tag{4.12}$$

Proof.

Case 1: $0 < \alpha < \infty$. The variational inequality defined in (4.12) is in fact a necessary and sufficient optimality condition of the convex program (4.11), since the Hessian matrix of $\phi(u,v)$with respect to u is symmetric positive definite.

Case 2: $\alpha = \infty$. $s(v)^T(u-H(v))$ is known as the gap function (Marcotte, 1985) which is always greater than or equal to zero, i.e., $s(v)^T u \geq s(v)^T H(v)$. Thus $H(v)$ is a solution (4.11) and vice versa. □

Proposition 4.1.2 *$H(v)$ has a fixed point $v \in \Omega$, i.e., $v \in H(v)$.*

Proof. Since $H(v)$ is an upper semicontinuous point-to-set mapping (continuous, if $0 < \alpha < \infty$), the result follows from Kakutani's fixed point theorem (1964). □

Proposition 4.1.3 *A vector v^* solves the variational inequality problem VIP if and only if it is a fixed point of the mapping H, i.e., $v^* \in H(v^*)$.*

Proof. See Dafermos (1982, Lemma 1). □

Proposition 4.1.4

$$f(v) \text{ is Lipschitz continuous on } \Omega \quad . \tag{4.13}$$

$$\partial f(v) = \mathrm{Co}_{u \in H(v)}[s(v) - \nabla s(v)(u - v) - \frac{1}{\alpha}\nabla_v \phi(u, v)]. \tag{4.14}$$

where u is the solution set of the subproblem (4.11).

$$\text{f(v) is differentiable at } v \text{ if } 0 < \alpha < \infty. \tag{4.15}$$

Proof. See Clarke (1975) for (4.13) and (4.14). For (4.15), note that $0 < \alpha < \infty$ ensures uniqueness of $H(v)$ by Danskin's theorem (1966). We then have that $f(v)$ is differentiable. □

Theorem 4.1.1 *$f(v) \geq 0$ for all $v \in \Omega$ and $f(v^*) = 0$, i.e., v^* solves (4.1), if and only if v^* solves VIP.*

Proof. For the first part, since $f(v)$ is the negative of the optimal value of subproblem (4.11) which is less or equal to zero (take $u = v$), it is clear that $f(v) \geq 0$ for every $v \in \Omega$.

For the second part, suppose that v solves *VIP*. Then by Proposition 4.1.3, we have $v \in H(v)$, which implies $f(v) = 0$.

Conversely suppose that $f(v) = 0$. It then follows from $f(v)$ that

$$s(v)^T(H(v) - v) = -\frac{1}{\alpha}\phi(H(v), v). \tag{4.16}$$

Since $s(v)^T(H(v)-v)+\frac{1}{\alpha}\phi(H(v),v)\leq 0$, we have

$$s(v)^T(H(v)-v)\leq -\frac{1}{\alpha}\phi(H(v),v). \tag{4.17}$$

Combining (4.16) and (4.17), we have $v\in H(v)$. By Proposition 4.1.3, v is then a solution of VIP. □

Proposition 4.1.5 *Assume that $s(v)$ is strongly monotone on Ω and $\nabla_u\phi(u,v)^T(u-v)\geq -\nabla_v\phi(u,v)^T(u-v)$. Suppose that v is not a solution of VIP. Then there exists $u\in H(v)$ such that $d=u-v$ is a descent direction for f at v, i.e.,*

$$f'(v;d)<0;$$

more precisely,

$$f'(v;d)\leq -f(v), \quad \text{if } \alpha=\infty, \tag{4.18}$$

$$f'(v;d)\leq -\lambda_{min}||d||_2^2, \quad \text{if } 0<\alpha<\infty. \tag{4.19}$$

Proof.

Case 1: $\alpha=\infty$. We have $f'(v;d)\leq -f(v)$ (see Marcotte, 1985).

Case 2: $0<\alpha<\infty$. We have

$$\begin{aligned} f'(v;d) &= -(u-v)^T\nabla s(v)(u-v) \\ &\quad +[s(v)-\tfrac{1}{\alpha}\nabla_v\phi(u,v)]^T(u-v) \\ &\leq -\lambda_{min}||d||_2^2+[s(v)+\tfrac{1}{\alpha}\nabla_u\phi(u,v)]^T(u-v) \\ &\leq -\lambda_{min}||d||_2^2-[s(v)+\tfrac{1}{\alpha}\nabla_u\phi(u,v)]^T(v-u) \\ &\leq -\lambda_{min}||d||_2^2 \qquad \text{by (4.12).} \quad \square \end{aligned}$$

If the form (4.8) or (4.9) of $\phi(u,v)$ is used, then the second condition of Proposition 4.1.5 is satisfied.

Proposition 4.1.6 *Assume that $s(v)$ is strongly monotone on Ω and the matrix $\nabla B(v)(H(v)-v)$ is positive definite on Ω with minimum eigenvalue β_{min}, if the form (4.10) of $\phi(u,v)$ is used. Let $d = u - v$ where $u \in H(v)$. If $d \neq 0$, then there exists $u \in H(v)$ such that d is a descent direction for f at v, i.e.,*

$$f'(v;d) < 0;$$

more precisely,

$$f'(v;d) \leq -f(v), \quad \text{if } \alpha = \infty, \tag{4.20}$$

$$f'(v;d) \leq -(\lambda_{min} + \frac{\beta_{min}}{2\alpha})||d||_2^2, \quad \text{if } 0 < \alpha < \infty. \tag{4.21}$$

Proof.

Case 1: $\alpha = \infty$. We have $f'(v;d) \leq -f(v)$ (see Marcotte, 1985).

Case 2: $0 < \alpha < \infty$. We have

$$\begin{aligned} f'(v;d) &= (u-v)^T[s(v) - (\nabla s(v) - \tfrac{B(v)}{\alpha})(u-v) \\ &\quad -\tfrac{1}{2\alpha}(u-v)^T\nabla B(v)(u-v)] \\ &= [s(v) + \tfrac{B(v)}{\alpha}(u-v)]^T(u-v) \\ &\quad -(u-v)^T\nabla s(v)(u-v) \\ &\quad -\tfrac{1}{2\alpha}(u-v)^T\nabla B(v)(u-v)(u-v) \\ &\leq -(u-v)^T\nabla s(v)(u-v) \\ &\quad -\tfrac{1}{2\alpha}(u-v)^T\nabla B(v)(u-v)(u-v) \\ &\leq -(\lambda_{min} + \tfrac{\beta_{min}}{2\alpha})||d||_2^2. \quad \square \end{aligned}$$

In practice we may expect that the form (4.10) provides a steeper descent direction.

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Proposition 4.1.7 *Assume that $\nabla s(v)$ is positive definite on Ω and that $\nabla_u \phi(u,v)^T(u-v) \geq -\nabla_v \phi(u,v)^T(u-v)$. Then there is only one stationary point v^* of f which is the unique solution of VIP, i.e.,*

$$f'(v^*; v-v^*) \geq 0 \text{ for all } v \in \Omega$$

$$\Rightarrow \quad v^* \text{ is the unique solution of } VIP.$$

Proof. First we prove that any stationary point v^* of f is a solution of VIP. Suppose $f'(v^*; v-v^*) \geq 0$ holds for all $v \in \Omega$. We have:

$$\sup_{u \in H(v^*)} (v-v^*)^T [s(v^*) - \nabla s(v^*)(u-v^*) - \frac{1}{\alpha}\nabla_v \phi(u,v^*)] \geq 0$$

Letting $v = u$ and rearranging, we obtain, from (4.12),

$$(u-v^*)^T \nabla s(v^*)(u-v^*) \leq [s(v^*) + \frac{1}{\alpha}\nabla_v \phi(u,v^*)]^T (u-v^*) \leq 0$$

which implies $u = v^*$ since $\nabla s(v)$ is positive definite on Ω. By Proposition 4.1.2, we conclude that v^* is a solution of VIP.

It is well known, however, that if $\nabla s(v)$ is positive definite on Ω, then VIP has a unique solution. Suppose that there are two stationary points. Then as in the first part, the two stationary points must be solutions of VIP, which is a contradiction. □

Proposition 4.1.8 *Assume that $0 < \alpha < \infty$. If v^* is a solution of VIP, then v^* is a stationary point of $f(v)$ over Ω.*

Proof. Since $0 < \alpha < \infty$ and (4.4) is satisfied, subproblem (4.11) is well defined and its solution is unique, i.e., $u = H(v^*) = v^*$. We have, for every $v \in \Omega$,

$$f'(v^*; v-v^*)$$

$$= (v-v^*)^T [s(v^*) - (\nabla s(v^*)(u-v^*) - \frac{1}{\alpha}\nabla_v \phi(u,v^*))]$$

$$= s(v^*)^T (v-v^*).$$

Since $s(v^*)^T(v-v^*) \geq 0$, we have $f'(v^*; v-v^*) \geq 0$. □

Proposition 4.1.9 *Assume that $s(v)$ is strongly monotone on Ω, $0 < \alpha < \infty$, the form (4.10) of $\phi(u,v)$ is used and $\nabla B(v)(H(v)-v)$ is positive semi-definite on Ω. Then v^* is a solution of VIP, if and only if v^* is a stationary point of $f(v)$ over Ω.*

Proof. In this case, subproblem (4.11) is well defined and its solution is unique. Suppose that v^* is a solution of VIP. We have $H(v^*) = v^*$ and

$$\begin{aligned} f'(v^*; v-v^*) &= (v-v^*)^T[s(v^*) - (\nabla s(v^*) - \tfrac{B(v^*)}{\alpha})^T(H(v^*)-v^*) \\ &\quad -\tfrac{1}{\alpha}(H(v^*)-v^*)^T\nabla B(v^*)(H(v^*)-v^*)] \\ &= s(v^*)^T(v-v^*) \geq 0 \ \text{ for all } \ v \in \Omega. \end{aligned}$$

Conversely, we have

$$f'(v^*; v-v^*) \geq 0 \ \text{ for all } \ v \in \Omega.$$

Letting $v = H(v^*)$ in the above expression and rearranging, we have

$$\begin{aligned} &(H(v^*)-v^*)^T\nabla s(v^*)(H(v^*)-v^*) \\ &+\frac{1}{\alpha}(H(v^*)-v^*)^T\nabla B(v^*)(H(v^*)-v^*)(H(v^*)-v^*) \\ &\leq [s(v^*) + \frac{B(v^*)}{\alpha}(H(v^*)-v^*)]^T(H(v^*)-v^*) \\ &\leq 0. \end{aligned}$$

This implies that $H(v^*) = v^*$, since $\nabla s(v^*)$ is positive definite on Ω and $\nabla B(v^*)(H(v^*)-v^*)$ is positive-semidefinite Ω. By Proposition 4.1.3, v^* is a solution of VIP. □

Note that the above two propositions (4.1.7) and (4.1.9) imply that if v^* is a stationary point of $f(v)$ over Ω, then $f(v) \geq f(v^*)$ by Theorem 4.1.1.

We now consider an interpretation of the condition

$$\nabla_u\phi(u,v)^T(u-v) \geq -\nabla_v\phi(u,v)^T(u-v) \tag{4.22}$$

with the following two propositions which help define $\phi(u,v)$.

Proposition 4.1.10 *Let* $\phi(u,v) = \theta(u) - \theta(v) - \nabla_v\theta(v)(u-v) : R^n \times R^n \to R^1$ *where*

1) $\theta(v)$ *is twice continuously differentiable;*

2) $\theta(u) - \theta(v) \geq \nabla_v\theta(v)(u-v);$ (4.23)

3) $\nabla_u\theta(u) - \nabla_v\theta(v) \geq \nabla_{vv}\theta(v)(u-v);$ (4.24)

4) $\theta(v)$ *is strictly convex.*

Then $\phi(u,v)$ *satisfies the conditions (4.2)-(4.7) and (4.22).*

Proof. 1) ensures (4.2). 2) ensures (4.3). 3) ensures (4.22), because

$$\begin{aligned}\nabla_v\phi(u,v) &= -\nabla_v\theta(v) - \nabla_{vv}\theta(v)(u-v) + \nabla_v\theta(v)\\ &= -\nabla_{vv}\theta(v)(u-v)\end{aligned}$$

so that

$$\begin{aligned}-\nabla_v\phi(u,v) &= \nabla_{vv}\theta(v)(u-v)\\ &\leq \nabla_u\theta(u) - \nabla_v\theta(v)\\ &= \nabla_u\phi(u,v).\end{aligned}$$

(4.5) is guaranteed by the definition of $\phi(u,v)$. 4) ensures (4.4). □

Proposition 4.1.11 *If* $\theta(u) = \frac{1}{2}||u||_B^2$, *then*

$$\phi(u,v) = \frac{1}{2}(u-v)^T B(u-v)$$

where B *is a symmetric and positive definite matrix.*

Proof.

$$\begin{aligned}\phi(u,v) &= \tfrac{1}{2}||u||_B^2 - \tfrac{1}{2}||v||_B^2 - v^T B(u-v)\\ &= \tfrac{1}{2}u^T Bu - \tfrac{1}{2}v^T Bv - v^T Bu + v^T Bv\\ &= \tfrac{1}{2}u^T Bu + \tfrac{1}{2}v^T Bv - v^T Bu\\ &= \tfrac{1}{2}(u-v)^T B(u-v). \quad \square\end{aligned}$$

Following the above analysis, we are ready to introduce the following general descent framework for VIP.

Proposition 4.1.12 *Let $\{v^l\}$ be a sequence generated by the iteration*

$$v^{l+1} = v^l + t_l d^l, \quad l = 0, 1, 2, ... \tag{4.25}$$

where $d^l = u^l - v^l$ ($u^l \in H(v^l)$) is a descent direction of f at v^l and $t_l \in [0, 1]$ is determined from

$$f(v^l + t_l d^l) = \min\{f(v^l + t d^l) \quad | \quad 0 \leq t \leq 1\}. \tag{4.26}$$

Suppose that Ω is compact and convex, the mapping $s(v) : \Omega \subset R^m \rightarrow R^m$ is continuously differentiable and $\nabla s(v)$ is positive definite on Ω. Furthermore, assume that $\nabla_u \phi(u, v) \geq -\nabla_v \phi(u, v)$ or $\nabla B(v)(H(v) - v)$ is positive definite on Ω. Then, for any starting point $v^0 \in \Omega$, the generated sequence $\{v^l\}$ lies in Ω and converges to a unique solution of the VIP.

Proof. Since the points v^l and $H(v^l) = v^l + d^l$ both belong to Ω and since $0 \leq t_l \leq 1$, it follows from the convexity of Ω that the sequence $\{v^l\}$ lies in Ω. By compactness of Ω, all limit points of the sequence $\{v^l\}$ stay in Ω.

To prove convergence of the algorithm, we use Zangwill's Convergence Theorem A (Zangwill, 1969). The assumption that Ω is compact ensures that Condition 1 of Convergence Theorem A holds. Condition 2 (the descent property) is ensured also by Propositions 4.1.5 and 4.1.6. Now consider Condition 3, i.e., upper semicontinuity of the algorithmic mapping

$$A(v^l) = M(v^l)D(v^l)$$

where

$M(v^l)$ is the line search mapping;
$D(v^l)$ is the direction mapping, i.e.,
$D(v^l) = (v^l, d^l) = (v^l, u^l - v^l)$ where $u^l \in H(v^l)$ and $d^l = u^l - v^l$.

We first study the direction mapping. Let $v^l \rightarrow v^\infty$ for a subsequence l and $d^l \rightarrow d^\infty$ for the subsequence l. Then

$$u^l \rightarrow u^\infty = d^\infty - v^\infty.$$

To prove that $D(v)$ is upper semicontinuous, it is sufficient to establish that $u^\infty \in H(v^\infty)$. Then, because $d^\infty = u^\infty - v^\infty$, we have

$$(v^\infty, d^\infty) \in D(v^\infty).$$

Since all u^l are feasible, u^∞ is also feasible. Moreover, by the definition of u^l, we have

$$[s(v^l) + \frac{1}{\alpha}\nabla_u \phi(u^l, v^l)]^T (u - u^l) \geq 0, \quad \text{for all} \quad u \in \Omega.$$

Since $\phi(u, v)$ is continuously differentiable by definition,

$$[s(v) + \frac{1}{\alpha}\nabla_u \phi(u, v)]$$

is upper semicontinuous.

Taking limits in the above inequality, we get

$$[s(v^\infty) + \frac{1}{\alpha}\nabla_u (u^\infty, v^\infty)]^T (u - u^\infty) \geq 0, \quad \text{for all} \quad u \in \Omega.$$

Then $u^\infty \in H(v^\infty)$ and $D(v)$ is upper semicontinuous.

It is well known that many search algorithms such as (4.26) induce the upper semicontinuity of $M(v)$. In this case, $A(v)$ is also upper semicontinuous.

We conclude that the algorithmic mapping is upper semicontinuous and consequently $f(v^l) \to 0$. This implies by Theorem 3.1 that $\{v^l\}$ is convergent to a solution of VIP which is unique since $s(v)$ is strongly monotone. All the other subsequences are also convergent to the solution. □

We note that the proposed algorithm is difficult to implement due to the fact that the evaluation of $f(v)$ involves an exact minimization, which is difficult to perform unless Ω has very special structure. Due to this fact, the exact line search is even more costly. An alternative approach for the line search would be to carry out an inexact line search, using for example an Armijo-type step rule (Ortega and Rheinboldt,

1970). If $\phi(u,v)$ is of the form (4.9), then the method is PBOM (see Fukushima, 1989). If $\phi(u,v)$ takes the form (4.10), we shall refer to the resulting algorithm as the Linearized Jacobi based method.

We close this section with the following equivalence results.

Proposition 4.1.13 *A solution v^* to VIP can be obtained by solving either one of the following equivalent problems:*

1) $s(v^*)^T(u-v^*) \geq 0$, for all $v \in \Omega$.

2) $\max_{v\in\Omega} \min_{u\in\Omega} s(v)^T(u-v) + \frac{1}{\alpha}\phi(u,v)$.

3) $\min -s(v)^T(u-v) - \frac{1}{\alpha}\phi(u,v)$

$$\text{s.t.} \quad v \in \Omega$$

$$u \in \arg\min s(v)^T(u-v) + \frac{1}{\alpha}\phi(u,v)$$

$$\text{s.t.} \quad u \in \Omega$$

4) $\max_{v\in\Omega} \min_{u\in\Omega} s(v)^T(u-v)$.

5) $\min -s(v)^T(u-v)$

$$\text{s.t.} \quad v \in \Omega$$

$$u \in \arg\min s(v)^T(u-v)$$

$$\text{s.t.} \quad u \in \Omega$$

We note that 3) and 5) are bilevel programming problems.

Proof.

1) $\Leftrightarrow$ 3). This is assured by Theorem 3.1.

2) $\Leftrightarrow$ 3). 2) is a reformulation of 3).

1) $\Leftrightarrow$ 4). This is assured by Theorem 3.1 when $\alpha = \infty$.

5) $\Leftrightarrow$ 4). 5) is only a reformulation of 4). □

From the above Proposition 4.1.13, we note that the variational inequality problem is a special case of the max-min problem while the max-min problem is a special case of the bilevel problem.

4.2 Various realizations of the general algorithmic framework

In this section, we consider various realizations of the general descent framework suggested by Proposition 4.1.12. This analysis serves as a bridge connecting some algorithms of both the iterative methods and optimization methods.

Sufficient conditions for convergence of the following algorithms are not presented here. The interested reader is referred to the corresponding references.

Proposition 4.2.1 *If* $0 < \alpha < \infty$ *and* $t_i = 1$, *and* $\phi(u,v)$ *is given by*

$$\phi(u,v) = \frac{1}{2}(u-v)^T B(v)(u-v),$$

then the algorithm corresponds to a family of iterative methods already discussed. As particular instances, we obtain

1. *the symmetrized Newton method where* $B(v) = \frac{\nabla s(v) + \nabla s(v)^T}{2}$ *(Hammond, 1984);*
2. *the Quasi-Newton methods where* $B(v)$ *is a symmetric positive definite matrix of approximation of* $\nabla s(v)$ *(Josephy, 1979);*
3. *the Linearized Jacobi method where* $B(v)$=$Diag(\nabla s(v))$;
4. *the projection method where* $B(v)$ *is a constant positive symmetric definite* $\overline{B}$ *(Dafermos, 1980, Bertsekas, 1982).*

Note that these four methods are not by nature descent algorithms.

Proposition 4.2.2 *If* $\nabla s(v)$ *is symmetric, then VIP is equivalent to a convex minimization program. If the forms (4.8), (4.9) or (4.10) of* $\phi(u,v)$ *are used and* $0 < \alpha < \infty$, *then the algorithm reduces to Migdalas's regularized Frank-Wolfe method.*

Proof. See Migdalas (1990). □

Proposition 4.2.3 *Assume that $\nabla s(v) = D$ where D is symmetric, positive definite and constant. If $\phi(u,v)$ is defined as (4.9) where B is diagonal with positive entries and furthermore Ω consists of one constraint and bounded variables, then the algorithm is equivalent to Dussault-Ferland-Lemaire's method (DFL).*

Proof. See Dussault, Ferland and Lemaire (1986). In this case, *VIP* represents the first-order optimality condition of a quadratic convex program such as

$$\min g(v) = \frac{1}{2}v^T Dv + c^T v$$

$$\text{s.t.} \quad v \in \Omega$$

where D is a symmetric and positive definite matrix, c is a vector and $s(v) = v^T D + c^T$.

In DFL, given a feasible v, a separable subproblem of the form:

$$(SS) \min \frac{1}{2}u^T Bu + C^T u$$

$$\text{s.t.} \quad u \in \Omega$$

where $C = c + (D - B)v$ is solved. Thus, we need only to show that this separable subproblem is equivalent to the subproblem defined in (4.11) if the form (4.9) of $\phi(u,v)$ is taken and $\alpha = 1$. Consider the corresponding objective function in (4.11):

$$s(v)^T(u - v) + \frac{1}{2}(u - v)^T B(u - v)$$

$$= (D^T v + c^T)(u - v) + \frac{1}{2}(u^T Bu + v^T Bv - 2u^T Bv)$$

$$= \frac{1}{2}u^T Bu + [c + (D - B)v]^T u + [\frac{1}{2}v^T Bv - s(v)^T v]$$

$$= \frac{1}{2}u^T Bu + C^T u + \frac{1}{2}v^T Bv - s(v)^T v$$

which is equivalent to the separable problem (SS). □

Proposition 4.2.4 *If $\alpha = 1$ and the form (4.9) of $\phi(u, v)$ is used, then the algorithm reduces to Fukushima's algorithm.*

Proof. Fukushima (1990) considers the form (4.9). □

Finally, it is interesting to note that the formulation of the second approach which Pang ([229], 1990) proposed to solve VIP has a certain relationship with our general equivalent optimization formulation in terms of the structure. His model is stated as follows:

$$\min (H(v) - v)^T (H(v) - v)$$

$$s.t. \quad v \in \Omega$$

$$H(v) = \arg\min s(v)^T (u - v) + \frac{1}{2}(u - v)^T (u - v)$$

$$s.t. \quad u \in \Omega$$

where Ω is polyhedral. A descent method is developed based on Newton's method.

4.3 An inexact method

As indicated in Proposition 4.1.12, a convex optimization subproblem must be solved at each iteration. In general, the problem can not be solved exactly. We will therefore, in this section, propose an inexact scheme that retains the global convergence property possessed by the previous algorithm. It is noted that the inexact scheme may offer a good trade-off between the amount of work required per iteration and the total number of iterations carried out. In the following, we restrict our analysis to the case where $0 < \alpha < \infty$.

Define

$$\epsilon_l = \min\{\frac{1}{l}, \epsilon||d^l||_2^2\} \tag{4.27}$$

where ϵ_l is an error term at iteration l, d^l is a descent direction at iteration l and ϵ is a positive parameter.

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We have the following in an obvious manner:

1) $\epsilon_l \leq \epsilon ||d^l||_2^2$

2) $\epsilon_l \leq \frac{1}{l}$

3) $\lim_{l \to \infty} \epsilon_l = 0.$

Let $\hat{u}$ satisfy the inequality:

$$[s(v^l) + \frac{1}{\alpha} \nabla_u \phi(\hat{u}, v^l)]^T (v - \hat{u}) \geq -\epsilon_l, \quad \text{for all } v \in \Omega, \tag{4.28}$$

i.e., $\hat{u}$ is an ϵ_l-optimal solution of the subproblem (4.11).

Proposition 4.3.1 *Let $u = H(v^l)$ and $d^l = \hat{u} - v^l$. If $\phi(u, v)$ is strongly convex in u, i.e., there exists a positive constant γ_{min} such that*

$$[\nabla_u \phi(u', v) - \nabla_u \phi(u'', v)]^T [u' - u''] \geq \gamma_{min} ||u' - u''||_2^2, \tag{4.29}$$

and we have

$$||\hat{u} - u||_2 \leq \sqrt{\frac{\epsilon \alpha}{\gamma_{min}}} ||d^l||_2. \tag{4.30}$$

Proof. By definition, u satisfies

$$[s(v^l) + \frac{1}{\alpha} \nabla_u \phi(u, v^l)]^T (v - u) \geq 0 \text{ for all } v \in \Omega. \tag{4.31}$$

Letting $v = u$ in (4.28), $v = \hat{u}$ in (4.31), and adding (4.28) and (4.31), we obtain, after rearranging terms:

$$[\nabla_u \phi(\hat{u}, v^l) - \nabla_u \phi(u, v^l)]^T [u - \hat{u}] \geq -\epsilon_l \alpha$$

$$||\hat{u} - u||_2^2 \leq \frac{\epsilon_l \alpha}{\gamma_{min}} \quad \text{using (4.29)}$$

$$\leq \frac{\epsilon \alpha}{\gamma_{min}} ||d^l||_2^2 \quad \text{using} \quad 1) \text{ above.}$$

Hence

$$||\hat{u} - u||_2 \leq \sqrt{\frac{\epsilon \alpha}{\gamma_{min}}} ||d^l||_2. \quad \square$$

Proposition 4.3.2 *Assume that*

1) $0 < \alpha < \infty$ (4.32)

2) $\nabla_u \phi(u,v)^T (u - v) \geq -\nabla_u \phi(u,v)^T (u - v)$ (4.33)

3) $\nabla_u \phi$ *is strongly monotone in* u *on* Ω (4.34)

4) $\nabla_u \phi$ *is Lipschitz continuous in* u *on* Ω, *with Lipschitz constant* L, *i.e.,*

$$||\nabla_u \phi(u', v) - \nabla_u \phi(u'', v)||_2 \leq L||u' - u''||_2, \forall u', u'' \in \Omega \quad (4.35)$$

5) $$\epsilon + \lambda_{max}\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}} + L\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}} < \lambda_{min} \quad (4.36)$$

where λ_{min} *and* λ_{max} *are the minimum and maximum eigenvalues of* $\nabla s(v)$ *respectively. If* $d^l = \hat{u} - v^l \neq 0$ *where* $\hat{u}$ *satisfies (4.28), then* d^l *is a descent direction of* $f(v)$ *at* v^l, *i.e.,* $f'(v^l; d^l) < 0$.

Proof. Let $u = H(v)$. We have

$$\begin{aligned} f'(v^l; d^l) &= (\hat{u} - v^l)^T [s(v^l) - \nabla s(v^l)(u - v^l) - \tfrac{1}{\alpha}\nabla_u \phi(u, v^l)] \\ &= [s(v^l) - \tfrac{1}{\alpha}\nabla_u \phi(u, v^l)]^T (\hat{u} - v^l) \\ &\quad -(\hat{u} - v^l)^T \nabla s(v^l)(u - v^l) \\ &= A1 + A2 \end{aligned}$$

where

$$A1 = [s(v^l) - \frac{1}{\alpha}\nabla_u \phi(u, v^l)]^T (\hat{u} - v^l)$$

$$A2 = -(\hat{u} - v^l)^T \nabla s(v^l)(u - v^l)$$

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Consider *A*1 :

$$
\begin{aligned}
A1 &= [s(v^j) - \tfrac{1}{\alpha}\nabla_u\phi(u, v^j)]^T(\hat{u} - v^j) \\
&\le [s(v^j) + \tfrac{1}{\alpha}\nabla_u\phi(u, v^j)]^T(\hat{u} - v^j) \qquad \text{by (4.33)} \\
&= -[s(v^j) + \tfrac{1}{\alpha}\nabla_u\phi(\hat{u}, v^j)]^T(v^j - \hat{u}) \\
&\quad + \tfrac{1}{\alpha}[\nabla_u\phi(u, v^j) - \nabla_u\phi(\hat{u}, v^j)]^T(\hat{u} - v^j) \\
&\le \epsilon_i + \tfrac{1}{\alpha}[\nabla_u\phi(u, v^j) - \nabla_u\phi(\hat{u}, v^j)]^T(\hat{u} - v^j) \qquad \text{by (4.28)} \\
&\le \epsilon\|d^j\|_2^2 + \tfrac{1}{\alpha}L\|u - \hat{u}\|_2 \cdot \|d^j\|_2 \qquad \text{by (4.35)} \\
&\le (\epsilon + L\sqrt{\tfrac{\epsilon}{\gamma_{min}}\alpha})\|d^j\|_2^2. \qquad \text{by (4.30)}
\end{aligned}
$$

Now consider *A*2:

$$
\begin{aligned}
A2 &= -(\hat{u} - v^j)^T\nabla s(v^j)(u - v^j) \\
&= -(\hat{u} - v^j)^T\nabla s(v^j)(\hat{u} - v^j) - (\hat{u} - v^j)^T\nabla s(v^j)(u - \hat{u}) \\
&\le -\lambda_{min}\|d^j\|_2^2 - (\hat{u} - v^j)^T\nabla s(v^j)(u - \hat{u}) \\
&\le -\lambda_{min}\|d^j\|_2^2 + \|d^j\|_2 \cdot \|\nabla s(v^j)\|_2 \cdot \|\hat{u} - u\|_2 \\
&\le -\lambda_{min}\|d^j\|_2^2 + \lambda_{max}\|d^j\|_2\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}}\|d^j\|_2 \qquad \text{by (4.30)} \\
&= (-\lambda_{min} + \lambda_{max}\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}})\|d^j\|_2^2
\end{aligned}
$$

Thus

$$
\begin{aligned}
&f'(v^j; d^j) \\
&\le (\epsilon + L\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}} - \lambda_{min} + \lambda_{max}\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}})\|d^j\|_2^2 < 0. \qquad \text{by (4.36)} \quad \square
\end{aligned}
$$

Proposition 4.3.3 *Assume that*

1) $\phi(u,v) = \frac{1}{2}(u-v)^T B(v)(u-v)$;

2) $\nabla B(v)(u-v)$ *is positive definite with minimum eigenvalue* β_{min};

3) *(4.29) is satisfied;*

4) $B(v)$ *is a symmetric and positive definite matrix with maximum eigenvalue* θ_{max};

5) $0 < \alpha < \infty$;

6) $\epsilon + (\frac{\beta_{max}}{\alpha} - \lambda_{min} - \frac{\beta_{min}}{2\alpha})\sqrt{\frac{\epsilon\alpha}{\gamma_{min}}} < \lambda_{min} + \frac{\beta_{min}}{2\alpha}$.

If $d^l = \hat{u} - v^l \neq 0$ *where* $\hat{u}$ *satisfies (4.28), then* d^l *is a descent direction for* $f(v)$ *at* v^l, *i.e.,*

$$f'(v^l; d^l) < 0.$$

Proof. Let $u = H(v^l)$. We have:

$$\begin{aligned} f'(v^l; d^l) &= (\hat{u} - v^l)^T [s(v^l) - (\nabla s(v^l) - \tfrac{B(v^l)}{\alpha})(u - v^l) \\ &\quad -\tfrac{1}{2\alpha}(u - v^l)^T \nabla B(v^l)(u - v^l)] \\ &= [s(v^l) - (\nabla s(v^l) - \tfrac{B(v^l)}{\alpha})(u - v^l)]^T (\hat{u} - v^l) \\ &\quad -\tfrac{1}{2\alpha}[(u - v^l)^T \nabla B(v^l)(u - v^l)]^T (\hat{u} - v^l) \\ &= [s(v^l) + \tfrac{B(v^l)}{\alpha}(u - v^l)]^T (\hat{u} - v^l) \\ &\quad -(u - v^l)^T \nabla s(v^l)(\hat{u} - v^l) \\ &\quad -\tfrac{1}{2\alpha}[(u - v^l)^T \nabla B(v^l)(u - v^l)]^T (\hat{u} - v^l) \end{aligned}$$

$$
\begin{aligned}
&= [s(v^l) + \tfrac{B(v^l)}{\alpha}(\hat{u} - v^l) + \tfrac{B(v^l)}{\alpha}(u - \hat{u})]^T(\hat{u} - v^l) \\
&\quad -(\hat{u} - v^l)\nabla s(v^l)(\hat{u} - v^l) - (u - \hat{u})^T\nabla s(v^l)(\hat{u} - v^l) \\
&\quad -\tfrac{1}{2\alpha}[(\hat{u} - v^l)^T\nabla B(v^l)(u - v^l)]^T(\hat{u} - v^l) \\
&\quad -\tfrac{1}{2\alpha}[(u - \hat{u})^T\nabla B(v^l)(u - v^l)]^T(\hat{u} - v^l) \\
&\leq \epsilon_l - \lambda_{min}||d^l||_2^2 - \tfrac{1}{2\alpha}\beta_{min}||d^l||_2^2 \\
&\quad +(u - \hat{u})^T\tfrac{B(v^l)}{\alpha}(\hat{u} - v^l) - (u - \hat{u})^T\nabla s(v^l)(\hat{u} - v^l) \\
&\quad -\tfrac{1}{2\alpha}(u - \hat{u})^T\nabla B(v^l)(u - v^l)(\hat{u} - v^l) \\
&\qquad\qquad \text{by (4.28) and 2)} \\
&\leq (\epsilon - \lambda_{min} - \tfrac{1}{2\alpha}\beta_{min})||d^l||^2 \\
&\quad +(\tfrac{\ell_{max}}{\alpha} - \lambda_{min} - \tfrac{\beta_{min}}{2\alpha})||\hat{u} - u||_2 \cdot ||d^l||_2 \\
&\leq [\epsilon - \lambda_{min} - \tfrac{1}{2\alpha}\beta_{min} + (\tfrac{\ell_{max}}{\alpha} - \lambda_{min} - \tfrac{\beta_{min}}{2\alpha})\sqrt{\tfrac{\epsilon\alpha}{\gamma_{min}}}] \cdot ||d^l||_2^2 \\
&< 0. \quad \text{by 6)} \quad \square
\end{aligned}
$$

An important issue is how to efficiently find $\hat{u}$ satisfying (4.28). As we already know, if $\epsilon = 0$, then condition (4.28) is the optimality condition of the subproblem (4.11), i.e.,

$$\min \psi(u, v^l) \tag{4.37}$$

$$\text{s.t. } u \in \Omega. \tag{4.38}$$

Let the optimal solution be u^*. By convexity and differentiability of ψ, $\hat{u}$ is an ϵ-optimal solution to (4.38), i.e., $\psi(u^*) + \epsilon \geq \psi(\hat{u})$.

Motivated by the network equilibrium problem where Ω is polyhedral and for which a linear program can easily be solved, we use the

linear approximation method (Frank-Wolfe) to solve (4.37) and (4.38).

Now consider the linear approximation problem for a given $\hat{u}$

$$\Phi = \min \nabla_u \psi(\hat{u}, v^l)(u - \hat{u})$$

$$\text{s.t. } u \in \Omega, \tag{4.39}$$

whose optimal solution is $\overline{u}$ and where $\nabla_u \psi(\hat{u}, v^l) = s(v^l) + \frac{1}{\alpha}\nabla_u \phi(\hat{u}, v^l)$. We assume that each subproblem of the Frank-Wolfe method is well defined. If $\Phi \geq -\epsilon_l$, then $\hat{H}(v^l) = \hat{u}$. Otherwise $d = \overline{u} - \hat{u}$ is a descent direction of $\psi(u, v^l)$ at $\hat{u}$. Then find a stepsize α^* such that

$$\alpha^* = \arg \min_{0 \leq \alpha \leq 1} \{\psi(\hat{u} + \alpha d, v^l)\} \tag{4.40}$$

Let $\hat{u} = \hat{u} + \alpha^* d$. Repeat the above procedure again. From the above discussion, we can see that subproblem (4.11) is solved by performing a sequence of linear programs. Now we are ready to present the inexact algorithm A for the VIP as follows.

Algorithm A

[step 1]	[Initialization]	Find $v^0 \in \Omega$ and set $l = 0$;
[step 2]	[Subproblem]	
[step 2.1]	[Initialization]	Let $u^0 = v^l$ and set $k = 0$;
[step 2.2]	[Linear program]	$\Phi = \min_{u \in \Omega} \nabla_u \psi(u^k, v^l)(u - u^k)$; Let the optimal solution be $\overline{u}$.
[step 2.3]	[Test]	Let $\epsilon_l = \min\{\frac{1}{l}, \epsilon \|d^l\|_2^2\}$. If $\Phi \geq -\epsilon_l$, then $\hat{u} = u^k$. Goto Step 3; else $d = \overline{u} - u^k$;
[step 2.4]	[Line search]	$\alpha^* = \arg\min_{0 \leq \alpha \leq 1} \{\psi(u^k + \alpha d, v^l)\}$;
[step 2.5]	[Update]	$u^{k+1} = u^k + \alpha^* d, \quad k = k + 1$. Goto Step 2.2;
[step 3]	[Descent direction]	$d^l = \hat{u} - v^l$;
[step 4]	[Stopping condition]	If $\|d^l\|_2 \approx 0$, then stop. The inexact solution is $\hat{u}$.
[step 5]	[Line search]	$\alpha_l = \arg\min_{0 \leq \alpha \leq 1} f(v^l + \alpha d^l)$;
[step 6]	[Update]	$v^{l+1} = v^l + \alpha_l d^l$; $l = l + 1$. Goto Step 2.

Proposition 4.3.4 *Assume that Ω is compact and convex, the mapping $s(v) : \Omega \subset R^m \to R^m$ is continuously differentiable, $\nabla s(v)$ is positive definite on Ω and, either $\nabla_u \phi(u,v)^T(u-v) \geq -\nabla_v \phi(u,v)^T(u-v)$ or $\nabla B(v)(u-v)$ is positive definite on Ω. If conditions in Propositions 4.3.1, 4.3.2 and 4.3.3 are satisfied and the solution of the linear program in step 2.2 exist, then from an arbitrary point $v^0 \in \Omega$, the sequence $\{v^l\}$ generated by the Algorithm A lies in Ω and converges to the unique solution of VIP.*

Proof. The proof will be similar to that of Proposition 4.1.12, if we can show that [step 2] in Algorithm A is well defined and convergent. In fact, the method for solving the subproblem in [step 2] is the application of the Frank-Wolfe method. Since $\Phi(u,v)$ is continuously differentiable and convex on u and Ω is closed and bounded, then according to Martos (1975), the sequence $\{u^k\}$ generated in [Step 2] converges globally to an ϵ-optimal solution to (4.37) and (4.38). □

If $\nabla s(v)$ is symmetric for all $v \in \Omega$, then Algorithm A reduces to Migdalas's method (1990).

4.4 Conclusion

In this chapter, we developed a general descent framework for the monotone variational inequality VIP and analyzed its relationship with some existing algorithms. Based on the analysis, we provided an inexact algorithm and proved that it converges globally under certain conditions.

It should be noticed that a decomposition approach can be applied to Algorithm A. In the case of the network equilibrium problem, the linear program in Step 2.2 can be decomposed into a set of independent linear subprograms, one per origin-destination pair. This decomposability is well suited to parallel processing. An obvious way to implement a parallel version is to assign one of the linear subprograms to each processor. Notice that this feature is not shared by Marcotte and Dussault's algorithm (1989) which requires the solution of a sequence of linear programs in the primal-dual space of variables.

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A disadvantage of using the Algorithm A for the network equilibrium problem is that we may not have a solution of the linear subproblem since the cost of the links may be negative and the negative cost cycle may exist. This is an interesting research field.

Chapter 5

Theoretical Development for VIP_h

In this chapter, we consider the transformed variational inequality problem, i.e., to find $h^* \in \Omega_h$ such that

$$S(h^*)^T(h - h^*) \geq 0, \forall h \in \Omega_h \tag{5.1}$$

where Ω_h is a closed and convex nonempty set. We assume that $H_v = \{h|\delta h = v\}$ is nonempty for all $v \in \Omega$. We study a special reformulation of it which we refer to as the linearized Jacobi based optimization model ($LJBOM$) for VIP_h. A prevailing difficulty particular to VIP_h is that the mapping $\nabla S(h) = \delta^T \nabla s(v)\delta$ may be positive semidefinite and the convergence of descent methods in this case is not guaranteed. In particular, for the network equilibrium problem in the space of path (hyperpath or strategy) flows, $\delta^T\delta$ is singular. We extend the theory developed in Chapter 4 to demonstrate that even when $\delta^T\delta$ is singular, our proposed method is still a descent and convergent method. After this analysis, we look at a special realization of the methods which is called the linearized Jacobi method. This method is also one of the linearized methods. Then we give a proof of global convergence of the method under some sufficient conditions different from those proposed earliear.

In particular, our methods can be implemented more easily for VIP_h than for VIP for the network equilibrium problem.

CHAPTER 5. THEORETICAL DEVELOPMENT FOR VIP_N

This chapter is organized as follows. In Section 1, the required notations and definitions for $LJBOM$ are introduced. Next, the $LJBOM$ is formulated and its structure is analyzed. Section 2 deals with a class of descent methods and a proof of convergence for such algorithms. In Section 3, we give a proof of global convergence of the linearized Jacobi method (LJM). In Section 4, we make a comparison between the LJM and the variable projection method.

5.1 Equivalent formulations of VIP_h

We consider first the following basic proposition.

Proposition 5.1.1 *Let $B(h)$ be an $n \times n$ symmetric, continuous and positive definite matrix for all $h \in R^n$ and α a positive parameter. The following three problems are equivalent for a given h:*

$$\min\{S(h)^T(u-h) + \frac{1}{2\alpha}(u-h)^T B(h)(u-h) \quad : \quad u \in \Omega_h\} \quad (5.2)$$

$$\min\{||u - (h - \alpha B^{-1}(h)S(h))||^2_{B(h)} \quad : \quad u \in \Omega_h\} \quad (5.3)$$

$$H(h) = Proj_{\Omega_h, B(h)}(h - \alpha B(h)^{-1}S(h)) \quad (5.4)$$

where $H(h)$ is the unique projection of the point $h - \alpha B(h)^{-1}S(h)$ onto the convex set Ω_h with respect to the norm $\|\cdot\|_{B(h)}$.

For completeness, we recall the following theorem.

Theorem 5.1.1 *h solves the variational inequality VIP_h if and only if h is a fixed point of the mapping H, i.e., $h = H(h)$.*

We assume that in the following α is always a positive parameter. We now study the following formulation. Consider the following problem:

$$(LJBOM) \qquad \min \ f(h) \ \ s.t. \ h \in \Omega_h \quad (5.5)$$

where

$$f(h) = -S(h)^T(H(h)-h) - \frac{1}{2\alpha}(H(h)-h)^T B(h)(H(h)-h). \quad (5.6)$$

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Theorem 5.1.2 *Let the function $f : R^n \to R^1$ be defined by (5.6). Then $f(h) \geq 0$ for $h \in \Omega_h$ and $f(h^*)=0$, i.e., h^* solves the LJBOM in (5.5) if and only if h^* solves the VIP_h.*

Proof. For the first part, since $f(h)$ is the negative of the optimal value of problem (5.2) which is less or equal to zero due to $h \in \Omega_h$, it is clear that $f(h) \geq 0$ for every $h \in \Omega_h$.

For the second part, suppose that h solves VIP_h. Then, by Theorem 5.1.1, we have that $h = H(h)$, which along with (5.6) implies $f(h) = 0$. Conversely suppose that $f(h) = 0$. It then follows from (5.6) that

$$-S(h)^T(H(h) - h) = \frac{1}{2\alpha}(H(h) - h)^T B(h)(H(h) - h). \qquad (5.7)$$

However, since $H(h) = Proj_{\Omega_h, B(h)}(h - \alpha B(h)^{-1}S(h))$, it follows that

$$(h-\alpha B(h)^{-1}S(h)-H(h))^T B(h)(u-H(h)) \leq 0 \text{ for all } u \in \Omega_h. \qquad (5.8)$$

Putting $u = h$ in (5.8) and rearranging terms, we obtain

$$S(h)^T(h - H(h)) \geq \frac{1}{\alpha}(H(h) - h)^T B(h)(H(h) - h)) \qquad (5.9)$$

Because $B(h)$ is positive definite, considering (5.7) and (5.9), it follows that $h = H(h)$. Consequently, by Theorem 5.1.1 h is a solution of VIP_h. □

Note that the *LJBOM* is reduced to the *PBOM* if $B(h) = \overline{B}$, where $\overline{B}$ is a symmetric positive definite matrix. In what follows, we discuss some important characteristics of the *LJBOM*. First we note that f can be a differentiable function.

Proposition 5.1.2 *If the mapping $s(v) : \Omega \subset R^m \to R^m$ is continuous, then the function $f(h) : \Omega_h \subset R^n \to R^1$ defined by (5.6) is also*

continuous. Furthermore, if $s(v)$ is continuously differentiable, then f is continuously differentiable and its gradient is given by

$$\begin{aligned}\nabla f(\lambda) &= -\nabla_\lambda \psi(\lambda, H(\lambda)) \\ &= S(\lambda) - (\nabla S(\lambda) - \frac{B(\lambda)}{\omega})^T (H(\lambda) - \lambda) \\ &\quad -\frac{1}{2\omega}(H(\lambda) - \lambda)^T \nabla B(\lambda)(H(\lambda) - \lambda)\end{aligned} \tag{5.10}$$

where

$$\psi(\lambda, u) = S(\lambda)^T (u - \lambda) + \frac{1}{2\omega}(u - \lambda)^T B(\lambda)(u - \lambda) \tag{5.11}$$

$\nabla B(\lambda)$ in $R^n \times R^n \times R^n$ is the derivative of the matrix $B(\lambda)$ in $R^n \times R^n$.

Proof. If $s(v) : \Omega \subset R^m \to R^n$ is continuous, then it follows from $S(\lambda) = \delta^T s(v), v = \delta\lambda$ and (5.4) that the mapping $S(\lambda)$ and $H(\lambda)$ are continuous. Hence, by (5.6), the function f is also continuous. For the second half of the proposition, we know ψ is continuous. Since, by definition,

$$f(\lambda) = -\min\{\psi(\lambda, u) | u \in \Omega_\lambda\} \tag{5.12}$$

and the minimum of the right-hand side of (5.12) is uniquely obtained at $u = H(\lambda)$, it follows from Auslender (1976, ch.4, Theorem 1.7) that f is differentiable and $\nabla f(\lambda)$ is given by (5.10). □

Corollary 5.1.1 *If λ is optimal, then $\nabla f(\lambda) = S(\lambda)$.*

Proof. By (5.10), we can see that if λ is optimal, then $\lambda = H(\lambda)$ and therefore the result holds. □

Proposition 5.1.3 *Suppose that there is a symmetric positive definite matrix $B(\lambda)$ for all $\lambda \in \Omega_\lambda$ and a parameter $0 < \theta < 1$ such that*

$$\begin{aligned}&\|B(\lambda)^{-1}\{\alpha[S(\overline{\lambda}(\lambda)) - S(\lambda)] - B(\lambda)[\overline{\lambda}(\lambda) - \lambda]\}\|_{B(\lambda)} \\ &\leq \theta \|\overline{\lambda}(\lambda) - \lambda\|_{B(\lambda)}\end{aligned} \tag{5.13}$$

holds for all $\lambda \in \Omega_\lambda$ where

$$\overline{\lambda}(\lambda) = \arg\min\{\|\lambda - y\|_{B(\lambda)} | y \in H^* \cap \Omega_\lambda\}, \tag{5.14}$$

$H^* = \{\lambda | v^* = \delta\lambda\}$ and v^* is the unique solution of VIP. Then

$$\|H(\lambda) - \overline{\lambda}(\lambda)\|_{B(\lambda)} \leq \theta \|\lambda - \overline{\lambda}(\lambda)\|_{B(\lambda)}. \tag{5.15}$$

Proof. Due to the properties of $H(h)$ and $\overline{h}(h)$, we have

$$[S(h) + \frac{B(h)}{\alpha}(H(h) - h)]^T[\overline{h}(h) - H(h)] \geq 0$$

and

$$S(\overline{h}(h))^T[H(h) - \overline{h}(h)] \geq 0.$$

Adding the two inequalities, we get

$$[H(h) - \overline{h}(h)]^T B(h)[H(h) - \overline{h}(h)]$$

$$\leq [H(h) - \overline{h}(h)]^T\{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}.$$

Thus:

$$\|H(h) - \overline{h}(h)\|^2_{B(h)}$$

$$\leq \|H(h) - \overline{h}(h)\|_{B(h)} \cdot$$

$$\|B(h)^{-1}\{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}\|_{B(h)}.$$

According to (5.13), we have

$$\|H(h) - \overline{h}(h)\|^2_{B(h)}$$

$$\leq \theta\|H(h) - \overline{h}(h)\|_{B(h)}\|\overline{h}(h) - h\|_{B(h)}.$$

Dividing by $\|H(h) - \overline{h}(h)\|_{B(h)}$, we obtain the result. □

We now examine the requirement (5.13). Consider the following linear transformation

$$v - v^* = \delta(h - \bar{h}(h)),$$

where

$$\bar{h}(h) = \arg\min\{\|h - y\|_{B(h)} | y \in H^*\}$$

and $H^* = \{h | \delta h = v^*\}$ where v^* is a solution of the VIP.

Proposition 5.1.4 *When viewed as a linear mapping from $h - \bar{h}(h)$ to $v - v^*$, δ is one-to-one and onto, i.e., δ is invertible.*

Proof. According to Nomizu (Proposition 3.1, 1966), δ is one-to-one and onto if and only if $\delta(0) = 0$. Now suppose that $h = \bar{h}(h)$. We have

$$v = \delta h = \delta\bar{h}(h) = v^*.$$

Reversely, suppose that $v = v^*$. We obtain $v = \delta h = v^*$. Thus $h \in H^*$ and $\bar{h}(h) = h$. □

By the same method, we have the following two propositions.

Proposition 5.1.5 *When viewed as a linear mapping from $h - \bar{h}(h)$ to $v - v^*$, δ is one-to-one and onto, i.e., δ is invertible where $h \in \Omega_h$.*

Proposition 5.1.6 *When viewed as a linear mapping from $s(v) - s(v^*)$ to $S(h) - S(\bar{h}(h))$, δ^T is one-to-one and onto, i.e., δ is invertible.*

Proposition 5.1.7 (Bertsekas and Gafni, 1982) *Assume*

$$\bar{h}(h) = \arg\min\{\|h - y\|_{B(h)} \mid y \in H^*\}.$$

Then there exists a positive scalar η such that for all $h \in \Omega_h$,

$$\|h - \bar{h}(h)\|_{B(h)} \geq \eta\|h - \bar{h}(h)\|_{B(h)}. \tag{5.16}$$

Proposition 5.1.8 *The requirement (5.13) is satisfied if*

$$\|s(v') - s(v'')\|_2 \leq L_s\|v' - v''\|_2, \tag{5.17}$$

$$[s(v') - s(v'')]^T(v' - v'') \geq \lambda\|v' - v''\|_2^2 \tag{5.18}$$

and $B(h)$ is a symmetric positive definite matrix for all $h \in \Omega_h$ with

$$1 > \theta^2 \geq \omega - 2\alpha\lambda\gamma\eta + \alpha^2 L_s^2\beta^2 > 0$$

where

L_s is a positive parameter;

α is a positive parameter;

λ is the minimum eigenvalue of the symmetric part of $\nabla s(v)$ over a feasible region which is larger than 0;

β is defined in Proposition 2.2.6 to be $\|\delta\|_F^2\Lambda_{min}^{-1}$;

Λ_{max} is the maximum eigenvalue of $B(h)$;

Λ_{min} is the minimum eigenvalue of $B(h)$;

γ is defined in Proposition 2.2.6 to be $\|\delta\|_F^2\Lambda_{max}^{-1}$;

ω is a positive paremeter larger than 1.

Proof. Let

$$\begin{aligned} A0 &= \|B(\lambda)^{-1}\{\alpha[S(\overline{\lambda}(\lambda)) - S(\lambda)] - B(\lambda)[\overline{\lambda}(\lambda) - \lambda]\}\|^2_{B(\lambda)} \\ &= \alpha^2[S(\overline{\lambda}(\lambda)) - S(\lambda)]^T \cdot B(\lambda)^{-1}[S(\overline{\lambda}(\lambda)) - S(\lambda)] \\ &\quad -2\alpha[S(\overline{\lambda}(\lambda)) - S(\lambda)]^T B(\lambda)^{-1} B(\lambda)(\overline{\lambda}(\lambda) - \lambda) \\ &\quad +(\overline{\lambda}(\lambda) - \lambda)^T \cdot B(\lambda) B(\lambda)^{-1} B(\lambda)(\overline{\lambda}(\lambda) - \lambda) \\ &= A1 - A2 + A3 \end{aligned}$$

We have the following inequalities:

$$\begin{aligned} A1 &\le \alpha^2 \Lambda_{\min}^{-1}[S(\overline{\lambda}(\lambda)) - S(\lambda)]^T[S(\overline{\lambda}(\lambda)) - S(\lambda)] \quad \text{by (2.20)} \\ &\le \alpha^2 \Lambda_{\min}^{-1}\|\delta\|_F^2 \|s(v^*) - s(v)\|_2^2 \quad \text{by (2.23)} \\ &\le \alpha^2 \Lambda_{\min}^{-1}\|\delta\|_F^2 L_s^2 \|v^* - v\|_2^2 \quad \text{by (5.17)} \\ &\le \alpha^2 \Lambda_{\min}^{-1}\|\delta\|_F^2 L_s^2 \|\delta\|_F^2 \Lambda_{\min}^{-1}\|\bar{\lambda}(\lambda) - \lambda\|^2_{B(\lambda)} \quad \text{by (2.27)} \\ &= \alpha^2 L_s^2 \beta^2 \|\bar{\lambda}(\lambda) - \lambda\|^2_{B(\lambda)} \quad \text{by definition of } \beta \\ &\le \alpha^2 L_s^2 \beta^2 \|\overline{\lambda}(\lambda) - \lambda\|^2_{B(\lambda)} \quad \text{by definition of } \bar{\lambda} \text{ and } \overline{\lambda}. \end{aligned}$$

$$\begin{aligned} A2 &\ge 2\alpha[S(\overline{\lambda}(\lambda)) - S(\lambda)]^T(\overline{\lambda}(\lambda) - \lambda) \\ &= 2\alpha[s(v^*) - s(v)]^T(v^* - v) \\ &\ge 2\alpha\lambda\|v^* - v\|_2^2 \quad \text{by (5.18)} \\ &\ge 2\alpha\lambda\gamma\|\bar{\lambda}(\lambda) - \lambda\|^2_{B(\lambda)} \quad \text{by (2.28)} \\ &\ge 2\alpha\lambda\gamma\eta\|\overline{\lambda} - \lambda\|^2_{B(\lambda)}. \quad \text{by (5.16)} \end{aligned}$$

$$\begin{aligned} A3 &\ge (\overline{\lambda}(\lambda) - \lambda)^T \cdot B(\lambda)(\overline{\lambda}(\lambda) - \lambda) \\ &= \|\overline{\lambda}(\lambda) - \lambda\|^2_{B(\lambda)}. \end{aligned}$$

There exists a positive parameter ω larger than 1 such that

$$A3 \leq \omega ||\overline{A}(h) - h||^2_{B(h)}.$$

Thus

$$A0 \leq \overline{\theta}^2 ||\overline{A}(h) - h||^2_{B(h)} \leq \theta^2 ||\overline{A} - h||^2_{B(h)}$$

where

$$1 > \theta^2 \geq \overline{\theta}^2 = \omega - 2\alpha\lambda\gamma\eta + \alpha^2 L_s^2 \beta^2 > 0. \qquad \square$$

Proposition 5.1.9 *Assume that the mapping $s(v) : \Omega \subset R^m \to R^m$ is continuously differentiable and its Jacobian matrix $\nabla s(v)$ is positive definite and $\nabla B(h)(H(h) - h)$ is positive definite. If h is a stationary point of the problem (5.5), i.e.,*

$$\nabla f(h)(u - h) \geq 0 \quad \textit{for all} \quad u \in \Omega_h, \tag{5.19}$$

then h is a global optimal solution of the problem (5.5) and hence it solves the variational inequality (5.1).

Proof. We know that

$$(H(h) - h)^T \nabla S(h)(H(h) - h)$$

$$= (v_H - v)^T \nabla s(v)(v_H - v) \geq 0$$

where $v_H = \delta H(h)$ and $v = \delta h$. Suppose that h satisfies (5.19). Letting $u = H(h)$, we have

$$(S(h) - (\nabla S(h) - \frac{B(h)}{\alpha})^T (H(h) - h))$$

$$-\frac{1}{2\alpha}(H(h) - h)^T \nabla B(h)(H(h) - h))^T (H(h) - h) \geq 0 \tag{5.20}$$

$$(S(h) + \frac{B(h)}{\alpha}(H(h) - h))^T (H(h) - h) \geq$$

$$(H(h) - h)^T \nabla S(h)(H(h) - h) +$$

$$\frac{1}{2\alpha}((H(h) - h)^T \nabla B(h)(H(h) - h))^T (H(h) - h). \tag{5.21}$$

On the other hand, since $H(h) = Proj_{\Omega_h, B(h)}(h - \alpha B(h)^{-1}S(h))$, it follows from

$$(h - Proj_{\Omega_h, B(h)}(h))^T B(h)(u - Proj_{\Omega_h, B(h)}(h)) \leq 0 \text{ for all } u \in \Omega_h \quad (5.22)$$

that

$$(h - \alpha B(h)^{-1}S(h) - H(h))^T \frac{B(h)}{\alpha}(u - H(h)) \leq 0 \text{ for all } u \in \Omega_h. \quad (5.23)$$

In particular, putting $u = h$ in (5.23), we obtain

$$(S(h) + \frac{B(h)}{\alpha}(H(h) - h))^T (H(h) - h) \leq 0. \quad (5.24)$$

Thus by (5.21) and (5.24), we have

$$(H(h) - h)^T \nabla S(h)(H(h) - h)$$

$$+\frac{1}{2\alpha}((H(h) - h)^T \nabla B(h)(H(h) - h))^T (H(h) - h) \leq 0. \quad (5.25)$$

Therefore, it follows that the inequality (5.25) holds only if $h = H(h)$. By Theorem 5.1.1, this implies that h solves VIP_h and by Theorem 5.1.2, h is a global optimal solution of problem (5.5). □

An important consequence of this proposition is that any local optimal solution of (5.5) is also a global optimal solution of (5.5). Thus any algorithm that finds a local optimal solution can be used to solve (5.5).

Proposition 5.1.10 *Let h be a solution of (5.5) and suppose that h is an interior point of Ω_h. Then $S(h) = 0$.*

Proof. See (Kinderlehrer and Stampacchia, 1980, ch.1, Corollary 3.2) and Theorem 3.1. □

Proposition 5.1.11 *Let h be a solution of (5.5). Then $S(h)$ determines a hyperplane of support for Ω_h provided $S(h) \neq 0$.*

Proof. See (Kinderlehrer and Stampacchia, 1980, ch.1, Corollary 3.4) and Theorem 3.1. □

This observation is important since it justifies the use of projection based methods, such as the *LJBOM* and the *PBOM*, for solving VIP_h, since such methods always find solutions on a face or at an extreme point of Ω_h.

Proposition 5.1.12 *Suppose that v^* is the unique solution of VIP and Ω_h is convex. Then the solution set of (5.5) is convex.*

Proof. Suppose h^1 and h^2 are two solutions where $h^1 \neq h^2, h^1, h^2 \in \Omega_h$, each satisfying

$$v^* = \delta h^1 \tag{5.26}$$

$$v^* = \delta h^2 \tag{5.27}$$

Thus suppose that there is an α, $0 \leq \alpha \leq 1$, such that $h = \alpha h^1 + (1-\alpha)h^2$ is not in Ω_h. Multiplying (5.26) and (5.27) by α and $(1-\alpha)$ respectively, and adding them up, we have

$$\alpha v^* + (1-\alpha)v^* = \alpha\delta h^1 + (1-\alpha)\delta h^2$$

$$v^* = \delta(\alpha h^1 + (1-\alpha)h^2) = \delta h$$

which means h is a feasible solution and a contradiction results. □

Next, we present several equivalent formulations of (5.5) which are useful in understanding the structure of the VIP_h.

Proposition 5.1.13 *Suppose that $s(v)$ is positive definite on Ω and $\nabla B(h)(H(h)-h)$ is positive semidefinite. The following three formulations are equivalent:*

1) *LJBOM in (5.5).*
2) *A max-min formulation*

$$\max_{h\in\Omega_h} \min_{u\in\Omega_h} S(h)^T(u-h) + \frac{1}{2\alpha}(u-h)^T B(h)(u-h). \tag{5.28}$$

3) *A bilevel formulation*

$$\min_{h\in\Omega_h} -S(h)^T(u-h) - \frac{1}{2\alpha}(u-h)^T B(h)(u-h)$$

$$\min_{u\in\Omega_h} S(h)^T(u-h) + \frac{1}{2\alpha}(u-h)^T B(h)(u-h). \quad (5.29)$$

Proof. It is straightforward to show the result by using the definitions of the $LJBOM$, bilevel formulation (Bard and Falk, 1982) and max-min theory. □

The above proposition highlights the fact that the variational inequality problem studied here is a special case of the max-min problem and, furthermore, that the max-min problem is a special case of the bilevel problem.

5.2 Descent methods

Theorem 5.2.1 *Let the mapping $s(v) : \Omega \subset R^m \to R^n$ be continuously differentiable. If the Jacobian matrix $\nabla s(v)$ is and $\nabla B(h)(H(h)-h)$ are both positive definite on Ω and Ω_h respectively. Then for each $h \in \Omega_h$ the vector $d = H(h)-h$ satisfies the descent direction condition*

$$\nabla f(h)d < 0$$

for every $d \neq 0$. Moreover if $||\nabla B(h)(H(h)-h)|| \geq \eta \geq 0$, then

$$\nabla f(h)^T d \leq -\frac{\eta}{2\alpha}||d||_2^2.$$

Proof. From (5.10), we have

$$\begin{aligned}\nabla f(h)^T d &= (S(h) + \tfrac{B(h)}{\alpha}(H(h)-h))^T(H(h)-h) \\ &\quad -(H(h)-h)^T\nabla S(h)(H(h)-h) \\ &\quad -\frac{1}{2\alpha}((H(h)-h)\nabla B(h)(H(h)-h))^T(H(h)-h). \end{aligned} \quad (5.30)$$

Since

$$(h - \alpha B(h)^{-1}S(h) - H(h))^T B(h)(u - H(h)) \leq 0 \text{ for all } u \in \Omega_h,$$

by having $u = h$, we obtain

$$(S(h) + \frac{B(h)}{\alpha}(H(h) - h))^T (H(h) - h) \leq 0. \tag{5.31}$$

Combining (5.30) and (5.31), we obtain,

$$\begin{aligned} \nabla f(h)^T d &\leq -d^T \nabla S(h) d - \tfrac{1}{2\alpha} d^T \nabla B(h)(H(h) - h) d \\ &= -d^T \delta^T \nabla s(v) \delta d - \tfrac{1}{2\alpha} d^T \nabla B(h)(H(h) - h) d \\ &\leq -(v_H - v)^T \nabla s(v)(v_H - v) - \tfrac{\eta}{2\alpha} d^T d \\ &\leq -\tfrac{\eta}{2\alpha} \|d\|_2^2. \quad \square \end{aligned}$$

Theorem 5.2.1 is interesting because the descent direction is still guaranteed even though $\nabla S(h)$ is positive semidefinite. In the following, we present an algorithmic framework using the descent direction d.

Theorem 5.2.2 *Let $\{h^l\}$ be a sequence generated by the iteration*

$$f^{l+1} = h^l + t_l d^l, \quad l = 0, 1, 2, ... \tag{5.32}$$

where $d^l = H(h^l) - h^l$ and $t_l \in [0, 1]$ is determined from

$$f(h^l + t_l d^l) = \min\{f(h^l + t d^l) | \ 0 \leq t \leq 1\} \tag{5.33}$$

Let Ω_h be compact and convex, the mapping $s(v) : \Omega \subset R^m \to R^m$ be continuously differentiable , $\nabla s(v)$ be positive semidefinite on Ω and $\nabla B(h)(H(h) - h)$ be positive definite for all $h \in \Omega_h$ and let the condition (5.13) hold. Then, for any starting point $h^0 \in \Omega_h$, the generated sequence $\{h^l\}$ lies in Ω_h and converges to a solution of VIP_h.

Proof. Since the points h^l and $H(h^l) = h^l + d^l$ both belong to Ω_h and since $0 \leq t_l \leq 1$, it follows, from the convexity of Ω_h, that the sequence $\{h^l\}$ is contained in Ω_h. By compactness of Ω_h, all limit points of the sequence $\{h^l\}$ stay in Ω_h.

CHAPTER 5. THEORETICAL DEVELOPMENT FOR VIP_H

To prove the convergence of the algorithm, we use Zangwill's Convergence Theorem A (Zangwill, 1969). The assumption that Ω_h is compact ensures that Condition 1 of Convergence Theorem A holds. Condition 2 (the descent property) is ensured also by Theorem 5.2.1. Now consider Condition 3, i.e., that the algorithmic mapping is closed. The algorithmic mapping which is a point-to-set mapping is defined as

$$A(h^l) = M(h^l)D(h^l)$$

where

$M(h^l)$ is the line search mapping;
$D(h^l)$ is the direction mapping, i.e.,
$D(h^l) = (h^l, d^l) = (h^l, H(h^l) - h^l)$.

Since $d^l = H(h^l) - h^l$ is continuous with respect of h^l, whenever $s(v)$ is continuous, $D(h^l)$ is continuous.

It is well known that the search algorithm determined by (5.33) induces the continuity of $M(h)$. In this case, $A(h)$ is also continuous.

This concludes that the algorithmic mapping is upper semicontinuous, hence closed, and $f(h^l) \to 0$. This implies, by Theorem 5.1.2, that $\{h^l\}$ converges a solution of VIP_h and any subsequence (hence the entire sequence) converges to the solution. □

The proposed algorithm is difficult to implement mainly due to the difficulties of the direct expression of h in f. Therefore it is impossible to carry out (5.33) exactly.

If t is set to be equal to 1, then the algorithm corresponds to the projection method or the linearized Jacobian method, depending on whether B is constant or varies with h. The algorithm, however, is not in general of the descent type.

5.3 A linearized method

5.3.1 The linearized method

A general linearized method for solving $VIP_h(\Omega_h, S)$ may be stated as follows: Given a $h^l \in \Omega_h$, $h^{l+1} = H(h^l)$ is defined as a solution to a variational inequality subproblem $VIP_h(\Omega_h, S^l)$ where

$$S^l(h) = S(h^l) + \frac{B(h^l)}{\alpha}(h - h^l) \tag{5.34}$$

where $B(h^l)$ is a square matrix and α is a positive parameter.

This class of methods include the linearized Jacobi method, where $B(h^l)$ is the diagonal of the Jacobian matrix of $S(h^l)$, i.e., $\nabla S(h^l)$, and the projection method, where $B(h^l) = B$ is some fixed symmetric positive definite matrix.

In both cases, each subproblem is well defined and has been proved to be equivalent to solving a strict convex quadratic minimization problem, due to the properties of $B(h^l)$, i.e.,

$$\min \int_0^h S^l(x)dx \tag{5.35}$$

$$\text{s.t.} \quad h \in \Omega_h$$

or

$$\min \ (h - h^l)^T S(h^l) + (h - h^l)^T \frac{B(h^l)}{2\alpha}(h - h^l) \tag{5.36}$$

$$\text{s.t.} \qquad h \in \Omega_h.$$

5.3.2 The linearized Jacobi method (LJM)

Theorem 5.3.1 (Convergence Theorem) *Let Ω_h be a nonempty compact and convex set. Let $s(v)$ be continuous. Let $B(h)$ be a symmetric positive definite matrix. Then there exists a symmetric positive definite matrix $\overline{B}$ with $B(h) - \overline{B}$ positive semi-definite for all $h \in \Omega_h$. If there is a parameter $0 < \theta < 1$ such that*

$$||\overline{B}^{-1}\{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}||_{\overline{B}} \leq \theta||\overline{h}(h) - h||_{\overline{B}} \tag{5.37}$$

holds for all $h \in \Omega_h$ where

$$\overline{h}(h) = \arg\min\{||h - y||_{\overline{B}} | y \in H^* \cap \Omega_h, \} \tag{5.38}$$

and $H^* = \{h | v^* = \delta h\}$ and v^* is a unique solution, then the sequence of iterates $\{h^l\}$ generated by the linearized Jacobi method converges to a solution $\overline{h}$ of the variational inequality $VIP(\Omega_h, S)$ for any initial vector $h^1 \in \Omega_h$ with the estimate

$$||h^l - \overline{h}||_{\overline{B}} \leq \frac{(\theta+1)\theta^{l-1}}{(1-\theta)} ||h^1 - \overline{h}(h^1)||_{\overline{B}}. \tag{5.39}$$

Proof. The existence of $\overline{B}$ is obvious since $\overline{B}$ can be selected such that

$$\overline{B} = \lambda_{min} I$$

where λ_{min} is the minimum eigenvalue of $B(h)$ and I is a corresponding identity matrix.

We know that in this case at least one solution $\overline{h}$ exists. Note that the matrix $B(h)$ is a diagonal positive symmetric definite for all $h \in \Omega_h$. Thus each $H(h)$ is well defined. Since $H(h), \overline{h}(h)$ solve $VIP(\Omega_h, \overline{S}(h))$ where $\overline{S}(h) = S(h) + \frac{B(h)}{\alpha}(H(h) - h)$, and $VIP(\Omega_h, S)$ respectively, we have

$$[S(h) + \frac{B(h)}{\alpha}(H(h) - h)]^T [\overline{h}(h) - H(h)] \geq 0$$

$$S(\overline{h}(h))^T [H(h) - \overline{h}(h)] \geq 0$$

Adding the two inequalities, we get

$$[H(h) - \overline{h}(h)]^T \cdot B(h)[H(h) - \overline{h}(h)]$$

$$\leq [H(h) - \overline{h}(h)]^T \{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}.$$

Therefore:

$$||H(h) - \overline{h}(h)||^2_{\overline{B}}$$

$$\leq ||H(h) - \overline{h}(h)||_{\overline{B}} \cdot ||B^{-1}\{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}||_{\overline{B}}$$

$$\leq \theta ||H(h) - \overline{h}(h)||_{\overline{B}} ||\overline{h}(h) - h||_{\overline{B}}.$$

Dividing by $||H(h) - \overline{h}(h)||_{\overline{B}}$, there comes

$$||H(h) - \overline{h}(h)||_{\overline{B}} \leq \theta ||h - \overline{h}(h)||_{\overline{B}}. \tag{5.40}$$

We call (5.40) the contraction property. Let $k > l \geq 1$. From the contraction property and the triangle inequality, we obtain

$$\begin{aligned}
||h^k - h^l||_{\overline{B}} &\leq \textstyle\sum_{j=l}^{k-1} ||h^{j+1} - h^j||_{\overline{B}} \\
&\leq \textstyle\sum_{j=l}^{k-1} ||h^{j+1} - \overline{h}(h^j)||_{\overline{B}} + ||h^j - \overline{h}(h^j)||_{\overline{B}} \\
&\leq \textstyle\sum_{j=l}^{k-1} (\theta + 1) ||h^j - \overline{h}(h^j)||_{\overline{B}} \\
&\leq \textstyle\sum_{j=l}^{k-1} (\theta + 1) ||h^j - \overline{h}(h^{j-1})||_{\overline{B}} \\
&\leq \textstyle\sum_{j=l}^{k-1} (\theta + 1)\theta ||h^{j-1} - \overline{h}(h^{j-1})||_{\overline{B}} \\
&\leq \textstyle\sum_{j=l}^{k-1} (\theta + 1)\theta^{j-1} ||h^1 - \overline{h}(h^1)||_{\overline{B}} \\
&\leq (\theta + 1)\tfrac{\theta^l - \theta^k}{\theta(1-\theta)} ||h^1 - \overline{h}(h^1)||_{\overline{B}}.
\end{aligned}$$

Since $0 < \theta < 1$, it follows that $\{h^l\}$ is a Cauchy sequence and must therefore converge to, say, $\overline{h}$ due to the compactness assumption. Being a contraction,

$$H(\overline{h}) = \lim_{l \to \infty} H(H(h^l)) = \lim_{l \to \infty} H(h^{l+1}) = \overline{h}.$$

This shows that $\overline{h}$ is indeed a fixed point of H. By Theorem 3.1.1, $\overline{h}$ is a solution of $VIP(\Omega_h, S)$. Finally, letting $k \to \infty$, we arrive at the estimate, since $\lim_{k \to \infty} \theta^k = 0$. □

Note that the requirement (5.37) is a special case of that (5.13) when $\overline{B}$ is determined by the way given in the beginning of the proof above.

5.4 Discussion

Here we compare the variable projection method developed by Bertsekas and Gafni (1982) with our linearized Jacobi method.

For completeness, we first review their algorithm. The main idea in this scheme is to allow a change in the matrix B only when the algorithm makes satisfactory progress towards convergence. Given v^{l+1} the matrix B^{l+1} is either chosen arbitrarily from φ (a set of positive definite symmetric matrices) or else $B^{l+1} = B^l$ depending on whether the quantity

$$w_l = (v^{l+1} - v^l)^T B^l (v^{l+1} - v^l)$$

has decreased or not by a certain factor since the last time the matrix B was changed. They proved that the variable projection is convergent. A potential drawback of the algorithm is that they have to evaluate w_l at each iteration in addition to the evaluation of B^l. However, the only work that the linearized Jacobi method has to do at each iteration is the evaluation of B^l.

Chapter 6

The Network Equilibrium Problem

6.1 The network equilibrium problem

It is well known that the network equilibrium problem may be formulated as the VIP or VIP_h (Smith, 1979, 1983, Dafermos, 1980 and Bertsekas and Gafni, 1982). We want to consider the problem in the path flow space which is equivalent to VIP_h, i.e., find $h^* \in \Omega_h$ an equilibrium path flow, such that

$$S(h^*)(h - h^*) \geq 0, \quad \forall h \in \Omega_h \tag{6.1}$$

where:

h^* is an equilibrium path flow vector in R^n;
h is any feasible path flow vector in R^n;
$S(h^*)$ is the path cost vector in R^n corresponding to h^*;
Ω_h is a feasible region defined as follows:

$$\sum_{k \in K_i} h_k = g_i, \quad \forall\ i \in I \tag{6.2}$$

$$h_k \geq 0, \quad \forall k \in K_i, \quad i \in I; \tag{6.3}$$

I is a set of origin-destination pairs;
K_i is a set of paths connecting origin-destination pair i;
h_k is a kth path flow;

g_i is the demand of the origin-destination pair i.

In the algorithmic development, we should observe two things. First, in order to find $H(h)$, we have to solve the quadratic program defined in (6.4), (6.5) and (6.6)

$$\min \quad \sum_{i \in I} \sum_{k \in K_i} [(h_k - h_k^l) S_k(h^l) + \frac{B_k(h^l)}{2\alpha}(h_k - h_k^l)^2] \qquad (6.4)$$

$$\text{s.t.} \quad \sum_{k \in K_i} h_k = g_i, \quad \forall \ i \in I \qquad (6.5)$$

$$h_k \geq 0 \qquad \forall \ k \in K_i \text{ and } i \in I. \qquad (6.6)$$

If the LJBOM or the PBOM is used, the $B(h)$ or $\overline{B}$ has a block diagonal form, then the program can be decomposed into a collection of smaller independent quadratic programs, one for each an origin-destination pair i. Second, since the number of paths in K_i is very large, it is preferable to start with a subset of each K_i and augment this subset as necessary. Bertsekas and Gafni (1982), among others, use a similar restriction method.

Fukushima (1989) adopted the inexact Armijo-type steplength rule to solve (5.33) without specifying the selection of the parameters in the method. Our solution method is basically the same, but it exploits the properties of the function f. This function has only one "valley", as shown in the Proposition 5.1.12. Suppose we are at one side of the "valley". If we find that the objective increases at a given steplength, we try to reduce it by reducing the steplength by half and evaluate the objective again to see if it is reduced. If not, the steplength is reduced by half again. Specifically, for the current iteration, with h^l and d^l, the steplength t_l is determined as $t_l = (\frac{1}{2})^k$, where k is the smallest nonnegative integer such that

$$f(h^l + t_l d^l) < f(h^l).$$

We apply Theorem 5.2.2 with the inexact steplength rule described above and the above two observations to develop the following specific algorithm (Algorithm B) for the network equilibrium problem.

Algorithm B

[step 1]	[initialization]	Determine a h_k^0, for all $k \in K_i^0$, $i \in I$, $l = 1$;
[step 2]	[shortest path]	Compute the shortest path k_i based on v^{l-1} from h^{l-1} for $i \in I$;
[step 3]	[update other path cost]	Compute $S_k(h^{l-1}), k \in K_i^{l-1}$;
[step 4]	[solve the quadratic program]	Find $H(h^{l-1})$ by solving the set of quadratic programs for each origin-destination pair i;
[step 5]	[find a descent direction]	$d^{l-1} = H(h^{l-1}) - h^{l-1}$;
[step 6]	[take a full steplength]	$h^l = H(h^{l-1})$;
[step 7]	[evaluate the objective value]	Compute $f(h^{l-1})$;
[step 8]	[optimality test]	If $\|\|\mathcal{L}^t(h^{l-1})\|\| \leq \epsilon$, then stop. (Here ϵ is a predetermined positive parameter.); else
[step 9]	[Armijo rule]	
[step 9.1]	[update the arc volumes]	$v^l = \delta h^l$ and the $s(v^l)$;
[step 9.2]	[update the cost of paths]	Find $S(h^l)$;
[step 9.3]	[solve the quadratic program]	Compute $H(h^l)$ with h^l and $S(h^l)$;
[step 9.4]	[evaluate the objective value]	Compute $f(h^l)$ with $H(h^l), h^l$ and $S(h^l)$;
[step 9.5]	[test the steplength]	If $f(h^l) \geq f(h^{l-1})$, then $h^l = h^{l-1} + \frac{(h^l - h^{l-1})}{2}$ and goto [step 9]; else $l = l + 1$ and goto [step 2].

6.2 Implementation issues

6.2.1 Separable convex quadratic program

In view of the nature of the block diagonal form of $B(h^l)$, as discussed, the quadratic program(6.4), (6.5) and (6.6) can be decomposed into a collection of smaller, independent quadratic programs, one per $i = (p, q)$ pair, i.e.,

$$\min \quad \sum_{k \in K_i} [(h_k - h_k^l) S_k(h^l) + \frac{B_k(h^l)}{2\alpha}(h_k - h_k^l)^2] \tag{6.7}$$

$$\text{s.t.} \quad \sum_{k \in K_i} h_k = g_i \tag{6.8}$$

$$h_k \geq 0 \qquad \forall \ k \in K_i. \tag{6.9}$$

There have been several research papers regarding to the solution of the problem of the one-constraint and bounded variables (Kennington, 1980, Dussault et al., 1986, Pardalos et al., 1990). Our problem is a little different from them in that we do not have upper bounds on h. In the following, we develop an efficient algorithm for the specific programs by using the same idea as Kennington (1980).

Since the program is strictly convex, the Kuhn-Tucker (K-T) conditions are necessary and sufficient for the optimality. Let μ and β_k be the dual variables associated with constraints (6.8) and (6.9) respectively, $B_k(h^l)$ be $B_k, K_i = K$ and $g_i = g$. We have the following K-T conditions:

$$S_k(h^l) + \frac{B_k}{\alpha}(h_k - h_k^l) - \mu - \beta_k = 0 \tag{6.10}$$

$$h_k \beta_k = 0 \tag{6.11}$$

$$\beta_k \geq 0 \tag{6.12}$$

$$\sum_{k \in K_i} h_k = g_i$$

$$h_k \geq 0, \forall k \in K_i.$$

Then we have

$$h_k = \max\{\frac{\alpha}{B_k}(\mu - S_k(h^l)) + h_k^l, 0\} \tag{6.13}$$

$$\beta_k = \max\{S_k(h^l) - \frac{B_k}{\alpha}h_k^l - \mu, 0\} \tag{6.14}$$

For a given μ, (6.13) and (6.14) always satisfy (6.9) and (6.12) by definition.

Proposition 6.2.1 *(6.13) and (6.14) satisfy (6.10).*

Proof.

Case 1. If

$$\frac{\alpha}{B_k}(\mu - S_k(h^l)) + h_k^l > 0,$$

then, by (6.13), we have

$$h_k = \frac{\alpha}{B_k}(\mu - S_k(h^l)) + h_k^l$$

which implies that

$$S_k(h^l) + \frac{B_k}{\alpha}(h_k - h_k^l) - \mu = 0.$$

From (6.14), we have $\beta_k = 0$. Thus

$$S_k(h^l) + \frac{B_k}{\alpha}(h_k - h_k^l) - \mu - \beta_k = 0.$$

Therefore (6.10) is satisfied.

Case 2. If

$$\frac{\alpha}{B_k}(\mu - S_k(h^l)) + h_k^l \leq 0,$$

then, by (6.13), we have $h_k = 0$. According to (6.14), we have

$$\beta_k = S_k(h^l) - \frac{B_k}{\alpha}h_k^l - \mu.$$

Therefore we obtain the following

$$S_k(h^l) + \frac{B_k}{\alpha}(h_k - h_k^l) - \mu - \beta_k = 0$$

which implies (6.10). □

Proposition 6.2.2 *(6.13) and (6.14) always satisfies (6.11).*

Proof. (6.13) and (6.14) ensure (6.11) as a direct consequence. □

Therefore the only constraint we need to worry about is (6.8). Now we define

$$\Gamma(\mu) = \sum_{k \in K} h_k(\mu) = \sum_{k \in K} \max\{\frac{\alpha}{B_k}(\mu - S_k(h^l)) + h_k^l, 0\}.$$

We should find μ^* such that $\Gamma(\mu^*) = g$. It is clear that the function $\Gamma(\mu)$ is nondecreasing, monotonic and piecewise linear. Since $\Gamma(\mu)$ is a sum of $|K|$ components, the breakpoints for the piecewise linear function $\Gamma(\mu)$ occur at $|K|$ points, i.e., μ_k^0 for $k = 1, ..., |K|$ where

$$\mu_1^0 \leq \mu_2^0 \leq ... \leq \mu_{|K|}^0$$

without loss of generality. Thus we have

$\Gamma(\mu) = 0$, if $\mu \leq \mu_1^0$;

$\Gamma(\mu) = \sum_{k \in K} \max\{\frac{\alpha}{B_k}(\mu_{|K|}^0 - S_k(h^l)) + h_k^l, 0\}$, if $\mu = \mu_{|K|}^0$.

We now present an algorithm for obtaining the μ^* such that

$\Gamma(\mu^*) = g$.

The procedure consists of a binary search to bracket μ^* between two breakpoints, followed by a linear interpolation.

Algorithm For μ^*

[step 0]	[initialization]	Set $l = 1, r = \|K\|, g_{max} = \infty, g_{min} = 0$ and let $\mu^0_1 \leq \mu^0_2 \leq ... \leq \mu^0_{\|K\|}$ be the breakpoints;
[step 1]	[test for bracketing]	If $r - l = 1$, then goto [step 4]; otherwise set $m = \lfloor \frac{l+r}{2} \rfloor$ where $\lfloor x \rfloor$ is the greater integer $\leq x$;
[step 2]	[compute $\Gamma(\mu^0)$]	Set $\Gamma(\mu^0_m) = \sum_{k \in K} \max\{\frac{\alpha}{B_k}(\mu^0_m - S_k(h^l)) + h^l_k, 0\}$;
[step 3]	[update]	If $\Gamma(\mu^0_m) = g$, then terminate with $\mu^* = \mu^0_m$;
		If $\Gamma(\mu^0_m) > g$, then set $r = m, g_{max} = \Gamma(\mu^0_m), \mu_{max} = \mu^0_m$ and goto [step 1];
		If $\Gamma(\mu^0_m) < g$, then set $l = m, g_{min} = \Gamma(\mu^0_m), \mu_{min} = \mu^0_m$ and goto [step 1];
[step 4]	[interpolate]	Terminate with $\mu^* = \mu_{min} + \frac{\mu_{max} - \mu_{min}}{g_{max} - g_{min}}(g - g_{min})$.

Thus once μ^* is determined, we can use equation (6.13) to calculate h. It is interesting to note that the algorithm is in fact a one dimensional search, similar to a line search approach.

Chapter 7

Transit Equilibrium Assignment

7.1 Introduction

In this chapter we consider the formulation, analysis and algorithmic resolution of a transit equilibrium assignment problem ($TEAP$). The purpose of the $TEAP$ is to model the flow of passengers on a congested transit network, induced by the travel demands which are given by a fixed origin-destination matrix. The transit network that corresponds to the $TEAP$ consists of the underlying road highway network and the transit lines that use it.

The transit assignment problem has been focused for many years since in various cities of medium or large size there have been growing traffic congestions on the highway. It has been recognized that the transit system has provided a good alternative to reduce the traffic on the road. Therefore it is important to provide a nice transit service to the passengers. In order to maintain a suitable service, we have to understand how passengers travel in the transit network.

The problem, i.e., the transit assignment (equilibrium) problem, has been studied for more than one decade with many models proposed.

Two issues should be considered when we study these models. The first issue is the assumption about the passenger travel behaviour, while the second issue is related to whether the transit network is congested or not, i.e., if the network is congested, then some equilibrium conditions should be assumed. Different combinations of these two issues lead to various models. For the case of uncongested transit networks, see a good survey of the models given by Spiess and Florian (1989). In this case, several assumptions have been proposed. The all-or-nothing assignment is a traditional assignment model which assumes that the passengers select shortest pathes that are independent of passengers flows. The multipath assignment assumes that several pathes are found first and the passengers are assigned proportionally to the costs of the pathes. These methods does not consider the passenger behaviour at a bus stop carefully. In fact, to model the behaviour of the passengers waiting at a bus stop shared by several transit lines, all the models assume certain behaviours that determine the passenger's choice. These behaviours are called strategies. Spiess and Florian (1989) formulated a transit assignment model for an uncongested transit network based on the concept of the optimal strategy, as a linear programming and solved by a polynomial time algorithm. Therefore in their model, passengers always travel according to the optimal strategy, which is in fact a kind of multipath assignment. Nguyen and Pallottino (1985) characterized and studied in detail the notion of the strategy, by using a graph theoretical approach, and developed a concept of a hyperpath. A strategy corresponds a hyperpath, i.e., a subnetwork containing all arcs of the corresponding strategy. At a bus stop, passengers always board the first vehicle to arrive from the set of attractive lines characterizing the hyperpath at that stop. The demand at a node is then assigned to the hyperpath in proportion to the fixed frequencies of lines in the attractive set of the hyperpath.

Although this class of models can be useful and efficient to study transit systems where the impact of congestion effects on the route choice is low, they are unsuitable in the simulation of transit flows in some cities, such as Shanghai. High congestion levels may be so severe that they affect the passenger's route choices. This is the second issue to consider.

Gendreau (1984) and De Cea and Fernandez(1988) extended the *TEAP* to include the consideration of congestion effects not only on the in-vehicle cost but also on the waiting time. The first tends to formulate the waiting times with queuing theory models on a expanded network, while the second models the waiting time to be a direct function of the flow-dependent effective frequency; however, only a solution algorithm for a simplified version of the model is presented. Spiess and Florian (1989) gave a general version of a nonlinear model in which in-vehicle travel times (or generalized travel costs) are increasing functions of passenger flow (called "discomfort functions"). The main limitations of these approachs are that waiting times at stops are not affected by congestion and there are no cross effects of flows on the cost functions. A similar formulation was proposed by Nguyen and Pallottino (1985). They show that, by explicitly defining the flow and structures of the hyperpath, the standard equilibrium traffic assignment model can be adapted to transit networks by substituting the hyperpath flow space for the path flow space. As in Spiess and Florian's model, the waiting times are assumed constant. It is noted that this adaptation is carried in the hyperpath flow space not in the arc flow space. In this sense, the transit equilibrium assignment model is different from the traditional traffic equilibrium model.

Our approach is to adopt the concept of hyperpath in the *TEAP* formulation and apply it on a general network representation of the physical transit network. This general network, where each arc is assigned a frequency and a travel cost function, is a generalization of the traditional physical network as stated by Spiess and Florian (1989). A transformation matrix, i.e., an arc-hyperpath incidence matrix, from the hyperpath flow to the arc flow, is composed of elements with values between 0 and 1. Recall that the elements of an arc-path incidence matrix are restricted to be 0-1 values. The formulation of the waiting cost is also different from the existing models, in that the waiting cost is equal to the sum of a function of the flow independent frequencies and a function of the waiting and in-vehicle flows.

It is well known that the traffic equilibrium assignment and the *TEAP* can be formulated as variational inequalities. Three main ap-

proaches have been investigated for the solution of variational inequalities on a transportation network, i.e., the iterative approach, where the original problem is replaced by a sequence of optimization problems (Pang and Chan, 1982 and Florian and Spiess, 1982); the optimization approach, where the problem is formulated as a mathematical program (Nguyen and Dupuis, 1984, Marcotte, 1985 and Marcotte and Guèlat, 1988); the simplicial decomposition method where each subproblem is solved only in a restricted region (Pang and Yu, 1984, Lawphongpanich and Hearn, 1984).

Our solution method belongs to the first and third categories and resorts to the successive resolution of the linearized optimization problems. The methods are distinguished from one another according to the particular form of linearization used. The linearized Jacobi and projection methods are special instances of it. In this paper we will borrow some of the ideas presented in Dafermos (1980, 1983), Bertsekas and Gafni (1982) and Pang and Chan (1982). We will in particular show that strong monotonicity in the arc space (not in the hyperpath space !) is sufficient to obtain global convergence and uniqueness (in arc flow space) results for our model.

The main contribution of the chapter is a theoretical and computational investigation of the *TEAP*, including the new definition of waiting cost, using the linearized Jacobi and the projection methods in the hyperpath flow space.

The rest of the chapter is organized as follows. Section 7.2 presents the basic model notation and the formulation. In Section 7.3, we provide some theoretical results. In Section 7.4, we give a general linearized method with special attention to both the Jacobi and the projection methods. Section 7.5 is devoted to the study of the impact of model parameters on algorithmic convergence and uniqueness of the solution. Implementation and computational results are reported in Section 7.6. Finally, some conclusions are presented in Section 7.7.

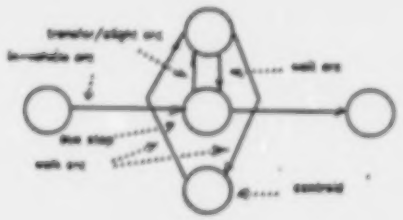

Figure 7.1: Representation of a transit stop in the general network

7.2 Notation and formulation

7.2.1 Basic notations and assumptions

Given a transit network, we have a set of distinct transit lines, transit stops and walk arcs, transfer and alight arcs and centroids where the passenger demand is generated and attracted. The network can be transformed into a general network where a typical stop node served by several transit lines is represented by a subgraph, in which each distinct wait arc is connected to a transit line. Such a subgraph is illustrated in Figure 7.1. This general network is defined as $G = \{N, A\}$ where $N = \{n | n = 1, 2, ..., |N|\}$ is the node set and $A = \{(t(a), h(a)) | a = 1, 2, ..., |A|, t, h \in N\}$ is the arc set where $t(a)$ and $h(a)$ are the tail and head respectively of the arc a. Let $s(v) = \{s_a(v) | a = 1, 2, ..., |A|\}$ be a vector of arc cost functions where v is the arc flow vector and let $f = \{f_a | a = 1, 2, ..., |A|\}$ be the set of arc frequencies. The general network has four kinds of arcs: walk arcs, wait arcs, on-board arcs and transfer/alight arcs represented by respective arc sets A^1, A^2, A^3 and A^4. Frequencies for the walk, in-vehicle and transfer/alight arcs can be infinite. Furthermore each travel arc is associated with exactly one transit line (one way) and there are no two nodes connecting more than two arcs and every wait arc is associated with exactly one transit line.

Now let us recall briefly the hyperpath concept. The hyperpath generalizes the concept of path; at each node of the network, a discrete variable specifies which outgoing arc will be travelled. A path may be viewed as a hyperpath for which the variable associated with a node can assume but one value, namely the index number of the successor

of the node in the path. Thus a given hyperpath k connecting a pair of nodes $(p,q) = i$ is a subgraph $G^k = \{N^k, A^k\}$. Let P_a^k denote the (conditional) probability that arc a is travelled, given that the flow on the hyperpath k has reached node $t(a)$. In the context of transit assignment we have

$$P_a^k = \frac{f_a}{\sum_{a' \in FS_k(t(a))} f_{a'}} \tag{7.1}$$

where $FS_k(t(a))$ is the forward star set of arcs at the node $t(a)$ in the hyperpath k. Let us define $\alpha^k_{t(a)}$ to be the proportion of one flow unit travelling hyperpath k at node $t(a)$; δ_a^k to be the proportion of one flow unit travelling hyperpath k on arc a. We then have the basic relationships for a hyperpath from origin node p to destination node q:

$$\alpha_p^k = 1$$

$$\alpha_q^k = 1$$

$$\delta_a^k = \alpha^k_{t(a)} P_a^k$$

$$\sum_{a' \in BS_k(h(a))} \alpha^k_{t(a)} P_a^k = \alpha^k_{h(a)}$$

where $BS_k(h(a))$ is the backward star set of arcs at the node $h(a)$ in the hyperpath k.

When a passenger is travelling over a given hyperpath at the node t (bus stop), he considers a set of attractive arcs corresponding to a set of transit lines (called attractive lines) and boards the first vehicle to arrive among the lines represented by those arcs. The waiting cost at the node includes a combination of two factors: the time that the passenger waits until one of the buses of those attractive arcs arrives, i.e., $\epsilon / \sum_{a \in FS_k(t)} f_a$, and the time that the passenger waits before boarding on the bus, i.e., $s_a(v)$ on the arc where ϵ is a parameter whose value depends on the distribution assumed for the buses interarrival time, i.e., 1 in case of exponential distribution or 0.5 in case of uniform distribution. The cost of a hyperpath k is defined as follows:

$$S_k = \sum_{a \in A} \delta_a^k [s_a(v) + w_{aa}] = \sum_{a \in A} \delta_a^k [s_a(v) + w_a P_a^k] \tag{7.2}$$

$$= S_k^1 + S_k^2 + S_k^3 + S_k^4$$

where $w_{ak} = \epsilon / \sum_{b \in FS_k(i)} f_b$, $w_a = \epsilon / f_a$ and $P_a^k = f_a / \sum_{b \in FS_k(i)} f_b$
$S_k^1 = \sum_{a \in A^1} \delta_{ak}^k s_a(v)$ is the walk cost on k
$S_k^2 = \sum_{a \in A^2} \delta_{ak}^k s_a(v) + \sum_{a \in A} \delta_{ak} w_{ak}$ is the wait cost on k
$S_k^3 = \sum_{a \in A^3} \delta_{ak}^k s_a(v)$ is the on-board cost on k
$S_k^4 = \sum_{a \in A^4} \delta_{ak}^k s_a(v)$ is the transfer/alight cost on k
$A = A^1 \bigcup A^2 \bigcup A^3 \bigcup A^4$
or in a vector notation, we obtain:

$$S = \delta^T [s(v) + w] \tag{7.3}$$

where δ is the arc-hyperpath matrix with elements δ_a^k, $s(v)$ is a vector of arc cost functions and w is a vector of w_a.

7.2.2 The structure of the cost functions

We consider a possible structure of the cost functions for the four kinds of arcs. As stated in Section 7.2.1, two attributes are associated with every arc, that is, $s_a(v)$ and $w_a = \epsilon / f_a$.

Walk arcs

We assume that the costs on the walk arcs are independent of flow and given by $s_a = \alpha_1 t_a$, where α_1 is a positive parameter and t_a is a constant walk time. f_a is set to be ∞ .

Wait arcs

The costs on the wait arcs are dependent on both the flows that correspond to waiting for the bus (v_a) and the direct flows (v_d) corresponding to through passengers that share the bus capacity with the waiting flow. The functional form chosen is as follows:

$$s_a(v_a, v_d) = \alpha_2 [(v_a + \beta_2 v_d)/K]^\rho \tag{7.4}$$

where α_2, β_2 and ρ are positive parameters and K is the bus capacity. According to the definition, we have $v_b = v_a + v_d$, where v_b is the in-vehicle flow (see Figure 7.2) and (7.4) may be rewritten as

$$s_a(v_a, v_b) = \alpha_2 [((1 - \beta_2) v_a + \beta_2 v_b)/K]^\rho \tag{7.5}$$

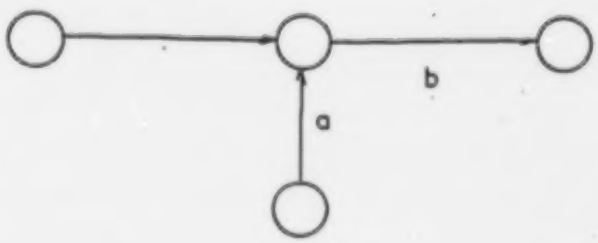

Figure 7.2: An example

with $0 \leq \beta_2 \leq 1$. Let f_a represent the bus frequency on arc b.

In-vehicle arcs

The costs of the in-vehicle arcs, t_b, and the discomfort costs depend on the direct flow, v_d, and wait flow, v_a, which is the total flow waiting for the bus. Thus we have the following relation as illustrated in Figure 7.2:

$$s_b(v_d, v_a) = \alpha_3 t_b + \beta_3[(v_d + \gamma_3 v_a)/K]^\rho \tag{7.6}$$

where α_3, β_3, γ_3 and ρ are the positive parameters associated with the in-vehicle arc. K is the capacity of the bus on arc a. Since $v_d = v_b - v_a$, we have

$$s_b(v_a, v_b) = \alpha_3 t_b + \beta_3[(v_b + (\gamma_3 - 1)v_a)/K]^\rho \tag{7.7}$$

We set f_b to be ∞.

Transfer and alight arcs

We assume that the arc costs are independent of arc flows and given by a constant walk time t_a, i.e., $s_a = \alpha_4 t_a$ and let the frequencies f_a be infinite, i.e., $f_a = \infty$.

7.2.3 Travel behaviour

The set of feasible hyperpath flows Ω_h is defined as follows:

$$\sum_{k \in K_i} h_k = g_i, \qquad i \in I \tag{7.8}$$

$$h_k \geq 0, \qquad k \in K_i, i \in I, \tag{7.9}$$

where h_k is the flow along hyperpath k, g_i is the travel demand from origin p to destination q, $(p,q) = i$, K_i is the set of hyperpaths from origin p to destination q, and I is the set of origin-destination pairs.

We assume that the passenger flows on the general network satisfy Wardrop's user-optimal equilibrium conditions (Wardrop,1952), i.e.,

$$S_{k^*} \begin{cases} = u_i^* & \text{if } h_{k^*} > 0 \\ \geq u_i^* & \text{if } h_{k^*} = 0 \end{cases} \tag{7.10}$$

where u_i^* is the minimum hyperpath cost from p to q and S_{k^*} the equilibrium hyperpath cost on k^*. It can be shown that the equilibrium conditions (7.10) can be alternatively stated as the following variational inequality:

$$VIP(\Omega_h, S): \qquad S(h^*)^T(h - h^*) \geq 0 \text{ for all } h \in \Omega_h \tag{7.11}$$

(or)

$$s(v^*)^T(v - v^*) + W^T(h - h^*) \geq 0 \quad \text{for all } v \in \Omega \text{ and } h \in \Omega_h \tag{7.12}$$

where h^* and v^* are the equilibrium hyperpath and arc flows respectively; Ω is a corresponding feasible arc flow set.

$$v_a = \sum_i \sum_k \delta_{ak} h_k \qquad (v = \delta h) \tag{7.13}$$

$$W_k = \sum_{a \in A} \delta_{ak} w_{ak} \qquad (W = \delta w). \tag{7.14}$$

Nguyen and Pallottino (1988) also derived the similar variational inequalities.

Theorem 7.2.1 *Assume that $s(v)$ is continuous and Ω is a compact and convex set. There is at least one solution h^* of $VIP(\Omega_h, S)$.*

Proof. Due to the assumption on s(v), we know that

$$S_k(h) = \sum_{a \in A} \delta_{ak}[s_a(\delta h) + w_{ak}]$$

is a positive continuous function of h. This ensures the existence of a solution h^* according to Smith(1979,1983). □

The following Theorem is stated in Nguyen and Pallottino (1988) without proof. For a completeness, we give a proof.

Theorem 7.2.2 *Consider the variational inequality* $VIP(\Omega_h, S)$. v^* *and* $W^T h^*$ *are unique if* $s(v)$ *is a positive strict monotone function, i.e.,*

$$[s(v') - s(v'')]^T [v' - v''] > 0 \tag{7.15}$$

Proof. Let us suppose that we have two solutions (v', h') and (v'', h''). Then we obtain

$$s(v')^T (v'' - v') + W^T (h'' - h') \geq 0 \qquad \text{and}$$

$$s(v'')^T (v' - v'') + W^T (h' - h'') \geq 0.$$

Adding them up, we have

$$[s(v') - s(v'')]^T (v' - v'') \leq 0.$$

Due to (7.15), we have $v' = v''$ and since h' and h'' satisfies the user-optimal equilibrium condition, we have $u^{rs} = u^{rs}$. Then

$$\sum_i \sum_k S_k(h') h'_k = \sum_i \sum_k S_k(h'') h''_k$$

$$\sum_a s_a(v') v'_a + \sum_i \sum_k W_k h'_k = \sum_a s_a(v'') v''_a + \sum_i \sum_k W_k h''_k \quad \Rightarrow$$

$$\sum_i \sum_k W_k h'_k = \sum_i \sum_k W_k h''_k$$

in compact form, we have

$$Wh' = Wh''. \qquad \square$$

7.3 Theoretical analysis

By definition, $B(h)$ is the diagonal of $\nabla S(h)$.

Proposition 7.3.1 $$\frac{\partial S_k}{\partial h_k} = \sum_{a \in A} \delta_{ak}^2 \frac{\partial s_a(v)}{\partial v_a}.$$

Proof. Since $S_k = \sum_{a \in A} \delta_{ak}(s_a(v) + w_{ak})$ and $v_a = \sum_i \sum_k \delta_{ak} h_k$, we have

$$\frac{\partial S_k}{\partial h_k} = \frac{\partial S_k}{\partial v_a}\frac{\partial v_a}{\partial h_k} = \sum_{a \in A} \delta_{ak}^2 \frac{\partial s_a(v)}{\partial v_a}. \qquad \square$$

Let u_p be the cost of hyperpath k from p to q, u_r be the cost of hyperpath $\overline{k}$ from r to q where $\overline{k}$ is a subhyperpath of k and f_p be the cumulative frequency of the hyperpath k at node p, i.e., $f_p = \sum_{b \in FS_k(p)} f_b$.

Then we have the following recursive formula ($a = (p, r)$) : for every $p \in N$,

$$S_k = u_p = \sum_{a \in FS_k(p)} (\frac{f_a}{f_p})(u_r + s_a(v) + w_{ak}),$$

$$u_q = 0.$$

The similar results can be found in Nguyen and Pallottino (1988). And obviously

$$\frac{\partial S_k}{\partial h_k} = \frac{\partial u_p}{\partial h_k} = \sum_{a \in FS_k(p)} (\frac{f_a}{f_p})^2 (\frac{\partial u_r}{\partial h_{\overline{k}}} + \frac{\partial s_a(v)}{\partial v_a}),$$

$$\frac{\partial u_q}{\partial h_k} = 0.$$

Theorem 7.3.1 (Convergence Theorem) *Let Ω_h be a nonempty compact and convex set. Let $s(v)$ be continuous. Let $B(h)$ be a symmetric positive definite matrix. Then there exists a symmetric positive definite*

matrix $\overline{B}$ with $B(h) - \overline{B}$ positive semi-definite for all $h \in \Omega_h$. If there is a parameter $0 < \theta < 1$ such that

$$||\overline{B}^{-1}\{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}||_{\overline{B}} \leq \theta||\overline{h}(h) - h||_{\overline{B}} \quad (7.16)$$

holds for all $h \in \Omega_h$ where

$$\overline{h}(h) = \arg\min\{||h - y||_{\overline{B}} | y \in H^* \cap \Omega_h, \} \quad (7.17)$$

and $H^* = \{h | v^* = \delta h\}$ and v^* is a unique solution, then the sequence of iterates $\{h^l\}$ generated by the linearized Jacobi method converges to a solution $\overline{h}$ of the variational inequality $VIP(\Omega_h, S)$ for any initial vector $h^1 \in \Omega_h$ with the estimate

$$||h^l - \overline{h}||_{\overline{B}} \leq \frac{(\theta + 1)\theta^{l-1}}{(1 - \theta)}||h^1 - \overline{h}(h^1)||_{\overline{B}}. \quad (7.18)$$

Proof. The proof is the same as that for VIP_h due to the fact that $W(h)$ is constant. Note that the structure of $S(h)$ here is a little different in that there is a constant vector w. □

7.4 Solution methods

7.4.1 The linearized Jacobi method (LJM)

The following describes the linearized Jacobi method (LJM) for the $TEAP$.

Algorithm LJM

[Step 1]	[initialization]	Let ϵ be a small positive parameter. Find $K_i^0 \in K_i, i \in I, h^0$ and v^0; Let $l = 1$.
[Step 2]	[shortest hyperpath]	For every $i \in I$, find a shortest hyperpath k_i, using $s(v^{l-1})$ and f. Output S_{k_i} and $\partial S_{k_i}/\partial h_{k_i}$ for all $k_i \in K_i^{l-1}, i \in I$.
[Step 3]	[hyperpath union]	For every i $\in I$, do $K_i^l = K_i^{l-1} \bigcup k_i$
[Step 4]	[cost evaluation]	Calculate cost and first derivative of the cost for all hyperpaths other than k_i. We have S_{k_i} and $\partial S_{k_i}/\partial h_{k_i}$ for $k_i \in K_i^{l-1}, i \in I$.
[Step 5]	[quadratic program]	For every $i \in I$, solve (6.7), (6.8), (6.9) and find h^l.
[Step 6]	[convergence test]	If $\|\|h^l - h^{l-1}\|\| \leq \epsilon$, then stop.
[Step 7]	[update]	$v^l = \delta h^l$ and $s(v^l)$ $l \leftarrow l + 1$ and return to step 2.

We recall that the shortest hyperpath problem has been studied in great depth by Spiess and Florian (1989) and Nguyen and Pallottino (1985, 1986) and can be solved efficiently by polynomial time algorithms.

At Step 6, one can use an alternative convergence measure at iteration l, CM^l, is:

$$CM^l = \sum_i \Delta h_i^l \Delta s_i^l / (g_i \, s_{min}^{i,l}) \tag{7.19}$$

where Δh_i^l is the portion of the demand that does not lie on the shortest paths of the ith pair at the end of iteration l; g_i is the demand for the ith pair; Δs_i^l is the difference between the largest and smallest

hyperpath cost for the ith pair at the end of iteration l; $s_{min}^{i,l}$ is the shortest hyperpath cost of the ith pair at iteration l.

It is clear that the CM^l is zero if and only if the solution is an equilibrium flow.

7.4.2 The projection method (PM)

If $B(h) = B$ is some fixed symmetric positive definite matrix, then algorithm LJM is the realization of a projection method (PM). Since the PM is a specialization of the LJM, we can easily conclude that Theorem 7.3.1 and the corresponding algorithm still apply for the PM with $B(h) = \overline{B} = B$.

From the computational point of view, we expect the LJM to perform better than the PM, because at each iteration the function $S(h^l)$ is better approximated. This has been confirmed in the computational experiments presented in Section 7.6.

7.5 Structural analysis of the cost function

The cost function parameters influence the model behaviour in mainly two respects:

1. The model's output results. In real applications, the parameters can be adjusted so that the results from the model can be best fitted to reality. This adjustment process is usually called model calibration.

2. Properties of the model. In our case, if some care is not taken, some assumptions would not be satisfied, especially the strong monotonicity assumption.

It is our hope that the range of parameters for which the calibrated functions satisfy the uniqueness and convergence conditions imposed on the model is large. This may not be the case.

Before discussing some properties of the model, we recall some basic results (see Ortega and Rheinboldt, 1970, for the proofs).

Proposition 7.5.1 *The arc cost function $s(v)$ is (strongly) monotone if its Jacobian matrix $\nabla s(v)$ is (positive definite) positive semidefinite over the compact and feasible region Ω.*

Proposition 7.5.2 *A square matrix A is positive semidefinite (positive definite) if and only if the following two equivalent statements are true:*

1. *for any nonzero vector x, $x \cdot Ax \geq 0$ (> 0);*
2. *all eigenvalues of the matrix $A + A^T$ are nonnegative (positive).*

If $-A$ is positive semidefinite (positive definite), then A is negative semidefinite (negative definite); otherwise it is said to be indefinite.

Proposition 7.5.3 *Let a symmetric matrix A have the block diagonal structure:*

$$\begin{pmatrix} A_{11} & 0 & & 0 \\ 0 & A_{22} & & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & & A_{nn} \end{pmatrix}.$$

where A_{ii}, $i=1,2,\ldots,n$, are square symmetric matrices. A is positive (negative) definite if A_{ii} are positive (negative) definite for all $i=1,\ldots,n$.

Proof. Straightforward. □

Let us now inspect in detail the cost structure of the general network. We have that

1. the cost of every wait arc depends on the flow not only on itself but also on its associated travel arc;
2. the cost of every in-vehicle arc depends on its own flow and also on the flow on the wait arc associated with it if any;
3. all other arc costs only depend on their own flow.

Wait arc (a)	In-vehicle arc (b)
$[(1-\beta_2)v_a+\beta_2 v_b]/K$	$\alpha_3 t_b + [v_b + (\gamma_3 - 1)v_a]/K$
$\alpha_2 = 1, \rho = 1$	$\beta_3 = 1, \alpha_3 = 1, \rho = 1$

Table 7.1: The cost structure

Without loss of generality, we can write $\nabla s(v)$ as

$$J = \begin{pmatrix} J_{12} & 0 & & 0 & & 0 \\ 0 & J_{34} & & 0 & & 0 \\ 0 & 0 & \cdot & 0 & & 0 \\ 0 & 0 & & J_{m-1,m} & & 0 \\ 0 & 0 & & 0 & \cdot & 0 \\ 0 & 0 & & 0 & & J_{n-1,n} \end{pmatrix}.$$

The submatrices $J_{i,i+1}$, $i = 1,..,m-1$, are the 2-dimensional square matrices with indices $1, 3, 5, ..., m-1$ corresponding to wait arcs and indices 2, 4, 6,...,m corresponding to travel arcs. The matrix $J_{i,i+1}, i = m+1, ..., n-1$, is in fact a 1-dimensional matrix, containing only one nonnegative element corresponding to the walk, in-vehicle and transfer/alight arcs. We will use a symmetric matrix $\overline{J} = (J + J^T)/2$ for the validation of the positive-definite matrix because of the relationship $x \cdot Jx = x \cdot 1/2(J + J^T)x = x\overline{J}x$. In view of Proposition 4.0.3, we only need to study $J_{i,i+1}, i = 1, ..., m-1$. As a special case, we consider the cost structure given in Table 7.1. for the wait arc a and the in-vehicle arc b.

The sub-Jacobian-matrix is given:

$$J_{ab} = \begin{pmatrix} (1-\beta_2)/K & \beta_2/K \\ (\gamma_3 - 1)/K & 1/K \end{pmatrix} \tag{7.20}$$

and its symmetric part is as follows:

$$\overline{J}_{ab} = \begin{pmatrix} (1-\beta_2)/K & (\beta_2 + \gamma_3 - 1)/2K \\ (\beta_2 + \gamma_3 - 1)/2K & 1/K \end{pmatrix}. \tag{7.21}$$

Let λ be the eigenvalues of $\overline{J}_{ab}$. We have

$$\lambda = [(2-\beta_2) \pm \sqrt{(2-\beta_2)^2 - 4(1-\beta_2) + (\beta_2 + \gamma_3 - 1)^2}]/K \tag{7.22}$$

Proposition 7.5.4 $\mathcal{J}_{ab}$ *is not negative definite.*

Proof.

$$\lambda_{max} = [(2-\beta_2) + \sqrt{(2-\beta_2)^2 - 4(1-\beta_2) + (\beta_2+\gamma_3-1)^2}]/K > 0$$

where $0 \leq \beta_2 \leq 1$ by definition. □

With a similar but more complicated approach, it is also possible to explore the monotonicity with respect to other parameters, in particular $\rho > 1$.

7.6 Implementation and computational experience

7.6.1 Implementation

A computer program written in Fortran 77 was developed on the SUN 3/50 (MC68020) workstations of the Center for Research on Transportation at the University of Montreal to implement the LJM and the PM algorithms.

7.6.2 Experimental design

Consider the 4-zone physical transit network given in Figure 7.3

The corresponding transformed network is given in Figure 7.4 and the specific information of the general network is presented in Table 7.2. The cost functions are defined as in Section 7.2.2. A brief description of three experiments is given in Table 7.3.

The results are analyzed in terms of the number of iterations (IT), path volume difference (PVD) and the convergence measure (CM).

Demand			
0	11	22	103
24	0	105	16
27	108	0	29
110	21	23	0

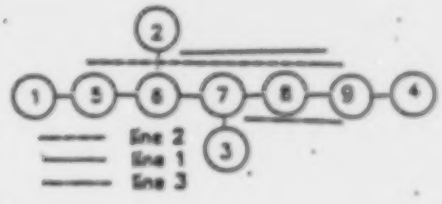

Figure 7.3: A 4-zone physical transit network

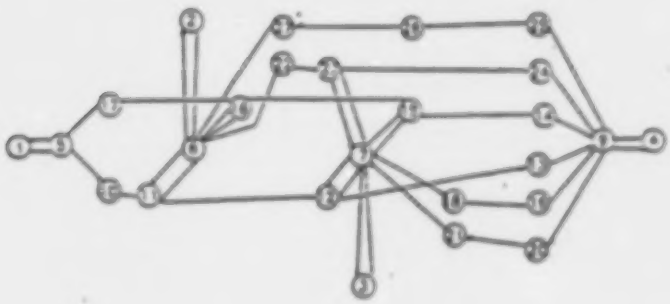

Figure 7.4: A 4-zone general network

arc type	arc	t_a	$1/f_a$
	(1,5)	2.	∞
	(2,6)	2.	∞
	(3,7)	2.	∞
walk	(4,9)	1.	∞
	(5,1)	2.	∞
	(6,2)	2.	∞
	(7,3)	2.	∞
	(9,4)	1.	∞
	(5,10)	0.	5.00
	(6,11)	0.	5.00
	(6,16)	0.	5.00
	(6,22)	0.	3.00
	(7,12)	0.	5.00
wait	(7,15)	0.	5.00
	(7,18)	0.	4.00
	(7,23)	0.	3.00
	(7,26)	0.	3.00
	(9,24)	0.	5.00
	(9,20)	0.	4.00
	(9,25)	0.	3.00
	(10,11)	2.	∞
	(11,12)	2.	∞
	(12,13)	3.	∞
	(14,15)	3.	∞
	(15,16)	2.	∞
travel	(16,17)	2.	∞
	(18,19)	3.	∞

(continued on the next page)

(continued from the previous page)			
	(20,21)	3.	∞
	(22,23)	2.	∞
	(23,24)	3.	∞
	(25,26)	3.	∞
	(26,27)	2.	∞
	(11,6)	3.	∞
	(12,7)	3.	∞
	(13,9)	3.	∞
	(15,7)	3.	∞
transfer/	(16,6)	3.	∞
alight	(17,5)	3.	∞
	(19,9)	3.	∞
	(21,7)	3.	∞
	(23,7)	3.	∞
	(24,9)	3.	∞
	(26,7)	3.	∞
	(27,6)	3.	∞

Table 7.2: Information on the general network

7.6.3 Computational results

Experiment 1 (EX.1)

The parameters selected for both PM and LJM are listed in Table 7.4 and the corresponding computational results are listed in Table 7.5. In fact, EX.1 is based on an asymmetric case problem with an affine cost function. In this case, the LJM is much more efficient than the PM, as noted by Nagurney (1984). The condition $[s(v^l) - s(v^{l-1})][v^l - v^{l-1}] > 0$

Experiment	Aims
1	Comparison between the PM and the LJM
2	Analysis of influence of the parameter α
3	Analysis of influence of the parameter ρ

Table 7.3: Description of three experiments

method	α_1	α_2	β_2	α_3	β_3	γ_3	α_4	K	α	$B(h)$	ρ
PM	1.0	1.0	0.5	1.0	1.0	1.5	1.0	50	0.88	1	1
LJM	1.0	1.0	0.5	1.0	1.0	1.5	1.0	50	0.88	*	1

*$B(h)$ varies at each iteration.

Table 7.4: The parameters for EX.1.

was satisfied for every pair of two consecutive points in the convergence sequence, even though the corresponding Jacobian matrix may not be necessarily positive definite here. Another observation is that the criteria PVD and CM are not consistent until iteration 5.

Experiment 2 (EX.2)

The parameters are unchanged with exception of α which is determined by trial and error for the LJM. The results of Table 7.6 show that the value 2.00 can be a good estimate of the optimal α in EX.2 and suggests that a substantial decrease in the number of iterations can result from such a choice.

Experiment 3 (EX.3)

In this experiment, α is set to 0.25 and ρ varies from 1 to 3. The results (Table 7.7) suggest that the computational effort increases with ρ. Indeed, S is better approximated for small ρ values.

	PM		LJM	
IT.	PVD	CM	PVD	CM
1	4.116	0.263×10^{0}	136.572	0.091×10^{0}
2	3.995	0.251×10^{0}	32.411	0.015×10^{0}
3	3.878	0.240×10^{0}	18.751	0.006×10^{0}
4	3.765	0.229×10^{0}	12.175	0.001×10^{0}
5	3.661	0.219×10^{0}	2.065	0.001×10^{0}
6	3.579	0.210×10^{0}	1.430	0.006×10^{-1}
7	3.499	0.202×10^{0}	1.016	0.004×10^{-1}
8	3.422	0.194×10^{0}	0.738	0.003×10^{-1}
9	3.342	0.186×10^{0}	0.566	0.002×10^{-1}
10	3.223	0.179×10^{0}	0.408	0.001×10^{-1}
11	3.144	0.172×10^{0}	0.308	0.001×10^{-1}
12	3.067	0.165×10^{0}	0.235	0.008×10^{-2}
.	. .	. .		
19	2.600	0.125×10^{0}	0.038	0.001×10^{-2}
.	. .			
100	0.354	0.013×10^{0}		

Table 7.5: Computational results for EX.1.

	$\alpha = 1.5$		$\alpha = 2.00$		$\alpha = 3.00$	
IT.	PVD	CM	PVD	CM	PVD	CM
1	181.60	0.063×10^{0}	212.800	0.051×10^{0}	261.594	0.033×10^{0}
2	50.378	0.004×10^{0}	58.658	0.002×10^{0}	113.178	0.008×10^{0}
3	6.190	0.009×10^{-1}	4.327	0.002×10^{0}	74.020	0.022×10^{0}
4	1.905	0.003×10^{-1}	0.668	0.008×10^{-1}	47.540	0.009×10^{-2}
5	0.891	0.001×10^{-1}	0.246	0.004×10^{-1}	24.434	0.013×10^{0}
6	0.489	0.001×10^{-1}	0.115	0.001×10^{-2}	14.200	0.004×10^{-1}
7	0.288	0.006×10^{-2}	0.056	0.008×10^{-2}	8.274	0.004×10^{0}
8	0.174	0.003×10^{-2}	0.027	0.003×10^{-2}	4.818	0.001×10^{-1}
9	0.107	0.002×10^{-2}	0.014	0.002×10^{-2}	2.805	0.001×10^{0}
10	0.066	0.001×10^{-2}	0.007	0.001×10^{-2}	1.633	0.006×10^{-2}

Table 7.6: Computational results for EX.2.

	$\rho = 1.$		$\rho = 2.$		$\rho = 3.$	
IT.	PVD	CM	PVD	CM	PVD	CM
1	43.882	0.209	156.029	1.720	212.616	5.295
2	36.745	0.124	60.872	0.664	59.491	1.580
3	29.224	0.071	39.860	0.367	19.899	0.749
4	14.186	0.047	21.379	0.215	12.988	0.449
5	12.501	0.031	11.946	0.177	8.720	0.801
6	11.083	0.019	9.341	0.151	8.689	0.708
7	5.745	0.013	9.199	0.137	9.630	0.656
8	5.043	0.011	9.075	0.125	9.324	0.583
9	4.665	0.008	8.963	0.116	8.943	0.520
10	4.319	0.007	8.860	0.108	8.555	0.467

Table 7.7: Computational results for EX.3.

7.7 Conclusion

In this chapter, we considered the transit equilibrium assignment problem with a new way of modeling the flow dependent waiting time. For this problem, it seems that the LJM performs very well and converges after a few iterations, which confirms our theoretical study. The new model behaves very well, especially in a congested transit network, where the waiting time increases with the passenger volume.

In the current model, the distribution of flow on a hyperpath is flow independent, which may be a drawback of the model presented. However, if it is to be flow dependent, then we have a nonlinear relationship , i.e., $v = \delta(h)h$ in place of the linear transformation. This class of models needs to be studied more in detail. Also, due to the large size of the physical transit network, we will in the future investigate data structures as well as easily parallellizable decomposition schemes aimed at reducing the computation time necessary to obtain a solution.

Chapter 8

Conclusion

In this thesis, we have studied the monotone variational inequality problem VIP in a systematic way. VIP consists in finding a solution $v \in \Omega$ which satisfies $s(v)^T(u-v) \geq 0$, for all $u \in \Omega$. The state of art of the research in theory and algorithms in the field is reviewed. An equivalent optimization problem of VIP is formulated. Based on it, a general descent framework for the problem is established with a proof of its convergence. Several realizations of the framework are also discussed to indicate its relationship with some existing algorithms. An inexact descent algorithm is also proposed. A similar analysis is applied to a transformed monotone variational inequality problem VIP_h where the cost function is only monotone. The developed framework then is applied to the network equilibrium problem where the cost structure is asymmetric. Some specific algorithms are presented with respect to the network structure. In what follows, we establish a model for transit equilibrium assignment. This model is then formulated as VIP_h and is solved with both the linearized Jacobi method and the projection method. Finally, numerical results are reported. The conclusions are given with possible research opportunities and general comments.

In short, the contributions of the thesis are:

1. A literature review of the VIP theory and algorithms is given;
2. A general descent framework for the VIP equivalent optimization problem is established with a detailed analysis;

3. An inexact method is developed for the VIP;
4. A global convergence proof is given for the linearized Jacobi method for VIP_A;
5. The similar analysis is applied to the VIP_A;
6. An efficient algorithm is developed for a special quadratic program;
7. A model for the transit equilibrium assignment is established and formulated as the VIP_A and is solved with the linearized Jacobi method.

As to research opportunities, we recommend the following several areas. First, the objective function in VIP may be investigated in terms of its structure. It would be interesting to look at other forms of $\phi(u, v)$ and even other forms of $f(v)$. Thus, various solution methods may be developed accordingly. The second, when $0 < \alpha < \infty$, $f(v)$ is differentiable, the K-T conditions of the formulation form a necessary and sufficient conditions for optimality. Based on these conditions, we can explore some feasible algorithms. Third, can we relax $B(v)$ to be asymmetric (instead of symmetric) and positive definite and then establish another formulation so that we have a Newton-type method?

In the application of VIP, we need to consider how the method works for very large scale problems. More computational results in transit equilibrium assignment are necessary to understand the behaviour of the model. As for the transit equilibrium assignment, it would be interesting to look at the problem with a variable δ. In this case, what are the convergence properties of the linearized Jacobi and the projection methods on the transformed variational inequality problem ?

In summary, the VIP has played a significant role in mathematical programming, and it is central to the modelling, computation and analysis of many equilibrium problems in economics, engineering, the social and physical sciences. The research in this subject is young and

fruitful. This thesis is only one example of work being done in the area. We are looking forward to seeing more developments in the future.

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